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QUANTUM FIELD THEORY

Dr Christoph Englert (University of Glasgow)
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Quantum Field Theory

Christoph Englert¹

These notes are a write-up of lectures given at the RAL school for High Energy Physicists, which took place at Warwick in 2014. The aim is to introduce the canonical quantisation approach to QFT, and derive the Feynman rules for a scalar field.

1 Introduction

Quantum Field Theory is a highly important cornerstone of modern physics. It underlies, for example, the description of elementary particles i.e. the Standard Model of particle physics is a QFT. There is currently no observational evidence to suggest that QFT is insufficient in describing particle behaviour, and indeed many theories for beyond the Standard Model physics (e.g. supersymmetry, extra dimensions) are QFTs. There are some theoretical reasons, however, for believing that QFT will not work at energies above the Planck scale, at which gravity becomes important. Aside from particle physics, QFT is also widely used in the description of condensed matter systems, and there has been a fruitful interplay between the fields of condensed matter and high energy physics.

We will see that the need for QFT arises when one tries to unify special relativity and quantum mechanics, which explains why theories of use in high energy particle physics are quantum field theories. Historically, Quantum Electrodynamics (QED) emerged as the prototype of modern QFT's. It was developed in the late 1940s and early 1950s chiefly by Feynman, Schwinger and Tomonaga, and has the distinction of being the most accurately verified theory of all time: the anomalous magnetic dipole moment of the electron predicted by QED agrees with experiment with a stunning accuracy of one part in 10¹⁰! Since then, QED has been understood as forming part of a larger theory, the Standard Model of particle physics, which also describes the weak and strong nuclear forces. As you will learn at this school, electromagnetism and the weak interaction can be unified into a single “electroweak” theory, and the theory of the strong force is described by Quantum Chromodynamics (QCD). QCD has been verified in a wide range of contexts, albeit not as accurately as QED (due to the fact that the QED force is much weaker, allowing more accurate calculations to be carried out).

As is clear from the above discussion, QFT is a type of theory, rather than a particular theory. In this course, our aim is to introduce what a QFT is, and how to derive scattering amplitudes in perturbation theory (in the form of Feynman rules). For this purpose, it is sufficient to consider the simple example of a single, real scalar field. More physically relevant examples will be dealt with.

¹SUPA, School of Physics and Astronomy, University of Glasgow.
Email: christoph.englert@glasgow.ac.uk
in the other courses. Throughout, we will follow the so-called canonical quantisation approach to QFT, rather than the path integral approach. Although the latter approach is more elegant, it is less easily presented in such a short course.

The structure of these notes is as follows. In the rest of the introduction, we review those aspects of classical and quantum mechanics which are relevant in discussing QFT. In particular, we go over the Lagrangian formalism in point particle mechanics, and see how this can also be used to describe classical fields. We then look at the quantum mechanics of non-relativistic point particles, and recall the properties of the quantum harmonic oscillator, which will be useful in what follows. We then briefly show how attempts to construct a relativistic analogue of the Schrödinger equation lead to inconsistencies. Next, we discuss classical field theory, deriving the equations of motion that a relativistic scalar field theory has to satisfy, and examining the relationship between symmetries and conservation laws. We then discuss the quantum theory of free fields, and interpret the resulting theory in terms of particles, before showing how to describe interactions via the $S$-matrix and its relation to Green’s functions. Finally, we describe how to obtain explicit results for scattering amplitudes using perturbation theory, which leads (via Wick’s theorem) to Feynman diagrams.

1.1 Classical Mechanics

Let us begin this little review by considering the simplest possible system in classical mechanics, a single point particle of mass $m$ in one dimension, whose coordinate and velocity are functions of time, $x(t)$ and $\dot{x}(t) = dx(t)/dt$, respectively. Let the particle be exposed to a time-independent potential $V(x)$. It’s motion is then governed by Newton’s law

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} = F(x),$$

where $F(x)$ is the force exerted on the particle. Solving this equation of motion involves two integrations, and hence two arbitrary integration constants to be fixed by initial conditions. Specifying, e.g., the position $x(t_0)$ and velocity $\dot{x}(t_0)$ of the particle at some initial time $t_0$ completely determines its motion: knowing the initial conditions and the equations of motion, we also know the evolution of the particle at all times (provided we can solve the equations of motion).

We can also derive the equation of motion using an entirely different approach, via the Lagrangian formalism. This is perhaps more abstract than Newton’s force-based approach, but in fact is easier to generalise and technically more simple in complicated systems (such as field theory!), not least because it avoids us having to think about forces at all.

First, we introduce the Lagrangian

$$L(x, \dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - V(x),$$

which is a function of coordinates and velocities, and given by the difference between the kinetic and potential energies of the particle. Next, we define the action

$$S = \int_{t_0}^{t_1} dt\, L(x, \dot{x}).$$
The equations of motion are then given by the *principle of least action*, which says that the trajectory \( x(t) \) followed by the particle is precisely that such that \( S \) is extremised \(^2\). To verify this in the present case, let us rederive Newton’s Second Law.

First let us suppose that \( x(t) \) is indeed the trajectory that extremises the action, and then introduce a small perturbation

\[
x(t) \to x(t) + \delta x(t),
\]

such that the end points are fixed:

\[
\begin{align*}
x'(t_1) &= x(t_1) \\
x'(t_2) &= x(t_2)
\end{align*}
\Rightarrow \delta x(t_1) = \delta x(t_2) = 0.
\]

This sends \( S \) to some \( S + \delta S \), where \( \delta S = 0 \) if \( S \) is extremised. One may Taylor expand to give

\[
S + \delta S = \int_{t_1}^{t_2} L(x + \delta x, \dot{x} + \delta \dot{x}) \, dt, \quad \delta \dot{x} = \frac{d}{dt} \delta x
\]

\[
= \int_{t_1}^{t_2} \left\{ L(x, \dot{x}) + \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \ldots \right\} \, dt
\]

\[
= S + \frac{\partial L}{\partial \dot{x}} \delta x \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right\} \delta x \, dt,
\]

where we performed an integration by parts on the last term in the second line. The second and third term in the last line are the variation of the action, \( \delta S \), under variations of the trajectory, \( \delta x \). The second term vanishes because of the boundary conditions for the variation, and we are left with the third. Now the Principal of Least Action demands \( \delta S = 0 \). For the remaining integral to vanish for arbitrary \( \delta x \) is only possible if the integrand vanishes, leaving us with the Euler-Lagrange equation:

\[
\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0.
\]

\(^2\)The name of the principle comes from the fact that, in most cases, \( S \) is indeed minimised.
If we insert the Lagrangian of our point particle, Eq. (2), into the Euler-Lagrange equation we obtain

\[ \frac{\partial L}{\partial x} = -\frac{\partial V(x)}{\partial x} = F \]
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{d}{dt} m\ddot{x} = m\ddot{x} \]
\[ \Rightarrow m\ddot{x} = F = -\frac{\partial V}{\partial x} \quad \text{(Newton’s law).} \] (8)

Hence, we have derived the equation of motion (the Euler-Lagrange equation) using the Principal of Least Action and found it to be equivalent to Newton’s Second Law. The benefit of the former is that it can be easily generalised to other systems in any number of dimensions, multi-particle systems, or systems with an infinite number of degrees of freedom, where the latter are needed for field theory.

For example, a general system of point particles has a set \( \{q_i\} \) of generalised coordinates, which may not be simple positions but also angles etc. The equations of motion are then given by

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \]

by analogy with the one-dimensional case. That is, each coordinate has its own Euler-Lagrange equation (which may nevertheless depend on the other coordinates, so that the equations of motion are coupled). Another advantage of the Lagrangian formalism is that the relationship between symmetries and conserved quantities is readily understood - more on this later.

First, let us note that there is yet another way to think about classical mechanics (that we will see again in quantum mechanics / field theory), namely via the Hamiltonian formalism. Given a Lagrangian depending on generalised coordinates \( \{q_i\} \), we may define the conjugate momenta \( p_i = \frac{\partial L}{\partial \dot{q}_i} \),

\[ \text{e.g. in the simple one-dimensional example given above, there is a single momentum } p = m\dot{x} \text{ conjugate to } x. \]

We recognise as the familiar definition of momentum, but it is not always true that \( p_i = m\dot{q}_i \).

We may now define the Hamiltonian

\[ H(\{q_i\}, \{p_i\}) = \sum_i \dot{q}_i p_i - L(\{q_i\}, \{\dot{q}_i\}). \]

As an example, consider again

\[ L = \frac{1}{2}m\dot{x}^2 - V(x). \]

It is easy to show from the above definition that

\[ H = \frac{1}{2}m\dot{x}^2 + V(x), \]
which we recognise as the total energy of the system. From the definition of the Hamiltonian one may derive (problem 1.1)
\[
\frac{\partial H}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{x}_i,
\]
which constitute Hamilton’s equations. These are useful in proving the relation between symmetries and conserved quantities. For example, one readily sees from the above equations that the momentum \( p_i \) is conserved if \( H \) does not depend explicitly on \( q_i \). That is, conservation of momentum is related to invariance under spatial translations, if \( q_i \) can be interpreted as a simple position coordinate.

### 1.2 Quantum mechanics

Having set up some basic formalism for classical mechanics, let us now move on to quantum mechanics. In doing so we shall use canonical quantisation, which is historically what was used first and what we shall later use to quantise fields as well. We remark, however, that one can also quantise a theory using path integrals.

Canonical quantisation consists of two steps. Firstly, the dynamical variables of a system are replaced by operators, which we denote by a hat. Secondly, one imposes commutation relations on these operators,
\[
[x_i, p_j] = i\hbar \delta_{ij} \tag{9}
\]
\[
[x_i, x_j] = [p_i, p_j] = 0 \tag{10}
\]

The physical state of a quantum mechanical system is encoded in state vectors \( |\psi\rangle \), which are elements of a Hilbert space \( \mathcal{H} \). The hermitian conjugate state is \( \langle \psi| = (|\psi\rangle)^\dagger \), and the modulus squared of the scalar product between two states gives the probability for the system to go from state 1 to state 2,
\[
|\langle \psi_1|\psi_2\rangle|^2 = \text{probability for } |\psi_1\rangle \rightarrow |\psi_2\rangle. \tag{11}
\]

On the other hand physical observables \( O \), i.e. measurable quantities, are given by the expectation values of hermitian operators, \( \hat{O} = \hat{O}^\dagger \),
\[
O = \langle \psi|\hat{O}|\psi\rangle, \quad O_{12} = \langle \psi_2|\hat{O}|\psi_1\rangle. \tag{12}
\]

Hermiticity ensures that expectation values are real, as required for measurable quantities. Due to the probabilistic nature of quantum mechanics, expectation values correspond to statistical averages, or mean values, with a variance
\[
(\Delta O)^2 = \langle \psi|(\hat{O} - O)^2|\psi\rangle = \langle \psi|\hat{O}^2|\psi\rangle - \langle \psi|\hat{O}|\psi\rangle^2. \tag{13}
\]

An important concept in quantum mechanics is that of eigenstates of an operator, defined by
\[
\hat{O}|\psi\rangle = O|\psi\rangle. \tag{14}
\]

Evidently, between eigenstates we have \( \Delta O = 0 \). Examples are coordinate eigenstates, \( \hat{x}|x\rangle = x|x\rangle \), and momentum eigenstates, \( \hat{p}|p\rangle = p|p\rangle \), describing a particle at position \( x \) or with momentum \( p \), respectively. However, a state vector cannot be simultaneous eigenstate of non-commuting
operators. This leads to the Heisenberg uncertainty relation for any two non-commuting operators \( \hat{A}, \hat{B} \),
\[
\Delta A \Delta B \geq \frac{1}{2} |\langle \psi | [\hat{A}, \hat{B}] |\psi \rangle |.
\]
Finally, sets of eigenstates can be orthonormalized and we assume completeness, i.e. they span the entire Hilbert space,
\[
\langle p' | p \rangle = \delta(p - p'), \quad 1 = \int d^3p \ |p\rangle \langle p|.
\]
As a consequence, an arbitrary state vector can always be expanded in terms of a set of eigenstates. We may then define the position space wavefunction
\[
\psi(x) = \langle x | \psi \rangle,
\]
so that
\[
\langle \psi_1 | \psi_2 \rangle = \int d^3x \langle \psi_1 | x \rangle \langle x | \psi_2 \rangle = \int d^3x \ \psi_1^*(x) \psi_2(x).
\]
Acting on the wavefunction, the explicit form of the position and momentum operators is
\[
\hat{x} = x, \quad \hat{p} = -i\hbar \nabla,
\]
so that the Hamiltonian operator is
\[
\hat{H} = \frac{\hat{p}^2}{2m} + V(x) = -\frac{\hbar^2 \nabla^2}{2m} + V(x).
\]
Having quantised our system, we now want to describe its time evolution. This can be done in different “pictures”, depending on whether we consider the state vectors or the operators (or both) to depend explicitly on \( t \), such that expectation values remain the same. Two extreme cases are those where the operators do not depend on time (the Schrödinger picture), and when the state vectors do not depend on time (the Heisenberg picture). We discuss these two choices in the following sections.

1.3 The Schrödinger picture

In this approach state vectors are functions of time, \( |\psi(t)\rangle \), while operators are time independent, \( \partial_t \hat{O} = 0 \). The time evolution of a system is described by the Schrödinger equation\(^3\),
\[
i\hbar \frac{\partial}{\partial t} \psi(x, t) = \hat{H} \psi(x, t).
\]
If at some initial time \( t_0 \) our system is in the state \( \Psi(x, t_0) \), then the time dependent state vector
\[
\Psi(x, t) = e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} \Psi(x, t_0)
\]
\(^3\)Note that the Hamiltonian could itself have some time dependence in general, even in the Schrödinger picture, if the potential of a system depends on time. Here we assume that this is not the case.
solves the Schrödinger equation for all later times $t$.
The expectation value of some hermitian operator $\hat{O}$ at a given time $t$ is then defined as

$$\langle \hat{O} \rangle_t = \int d^3x \, \Psi^*(x,t) \hat{O} \Psi(x,t),$$  \hspace{1cm} (22)

and the normalisation of the wavefunction is given by

$$\int d^3x \, \Psi^*(x,t) \Psi(x,t) = \langle 1 \rangle_t.$$  \hspace{1cm} (23)

Since $\Psi^* \Psi$ is positive, it is natural to interpret it as the probability density for finding a particle at position $x$. Furthermore one can derive a conserved current $\mathbf{j}$, as well as a continuity equation by considering

$$\Psi^* \times (\text{Schr.Eq.}) - \Psi \times (\text{Schr.Eq.})^*. $$  \hspace{1cm} (24)

The continuity equation reads

$$\frac{\partial}{\partial t} \rho = -\nabla \cdot \mathbf{j}$$  \hspace{1cm} (25)

where the density $\rho$ and the current $\mathbf{j}$ are given by

$$\rho = \Psi^* \Psi \quad \text{(positive)},$$

$$\mathbf{j} = \frac{\hbar}{2im} (\Psi^* \nabla \Psi - (\nabla \Psi^*) \Psi) \quad \text{(real)}. $$ \hspace{1cm} (26) \hspace{1cm} (27)

Now that we have derived the continuity equation let us discuss the probability interpretation of Quantum Mechanics in more detail. Consider a finite volume $V$ with boundary $S$. The integrated continuity equation is

$$\int_V \frac{\partial \rho}{\partial t} d^3x = -\int_V \nabla \cdot \mathbf{j} d^3x$$

$$= -\int_S \mathbf{j} \cdot d^2\mathbf{o}$$  \hspace{1cm} (28)

where in the last line we have used Gauss’s theorem. Using Eq. (23) the left-hand side can be rewritten and we obtain

$$\frac{\partial}{\partial t} \langle 1 \rangle_t = -\int_S \mathbf{j} \cdot d^2\mathbf{o} = 0.$$  \hspace{1cm} (29)

In other words, provided that $\mathbf{j} = 0$ everywhere at the boundary $S$, we find that the time derivative of $\langle 1 \rangle_t$ vanishes. Since $\langle 1 \rangle_t$ represents the total probability for finding the particle anywhere inside the volume $V$, we conclude that this probability must be conserved: particles cannot be created or destroyed in our theory. Non-relativistic Quantum Mechanics thus provides a consistent formalism to describe a single particle. The quantity $\Psi(x,t)$ is interpreted as a one-particle wave function.

1.4 The Heisenberg picture

Here the situation is the opposite to that in the Schrödinger picture, with the state vectors regarded as constant, $\partial_t |\Psi_H\rangle = 0$, and operators which carry the time dependence, $\hat{O}_H(t)$. This is the concept
which later generalises most readily to field theory. We make use of the solution Eq. (21) to the Schrödinger equation in order to define a Heisenberg state vector through

\[ \Psi(x,t) = e^{-\frac{i}{\hbar}H(t-t_0)}\Psi(x,t_0) \equiv e^{-\frac{i}{\hbar}H(t-t_0)}\Psi_H(x), \]

i.e. \( \Psi_H(x) = \Psi(x,t_0) \). In other words, the Schrödinger vector at some time \( t_0 \) is defined to be equivalent to the Heisenberg vector, and the solution to the Schrödinger equation provides the transformation law between the two for all times. This transformation of course leaves the physics, i.e. expectation values, invariant,

\[ \langle \Psi(t)|\hat{O}|\Psi(t)\rangle = \langle \Psi(t_0)|\hat{O}|\Psi(t_0)\rangle e^{\frac{i}{\hbar}H(t-t_0)} \equiv \langle \hat{O}_H(t)|\Psi_H\rangle, \]

with

\[ \hat{O}_H(t) = e^{\frac{i}{\hbar}H(t-t_0)}\hat{O} e^{-\frac{i}{\hbar}H(t-t_0)}. \]

From this last equation it is now easy to derive the equivalent of the Schrödinger equation for the Heisenberg picture, the Heisenberg equation of motion for operators:

\[ i\hbar \frac{d\hat{O}_H(t)}{dt} = [\hat{O}_H, \hat{H}]. \]

Note that all commutation relations, like Eq. (9), with time dependent operators are now intended to be valid for all times. Substituting \( \hat{x}, \hat{p} \) for \( \hat{O} \) into the Heisenberg equation readily leads to

\[ \frac{d\hat{x}_i}{dt} = \frac{\partial \hat{H}}{\partial \hat{p}_i}, \quad \frac{d\hat{p}_i}{dt} = -\frac{\partial \hat{H}}{\partial \hat{x}_i}, \]

the quantum mechanical equivalent of the Hamilton equations of classical mechanics.

### 1.5 The quantum harmonic oscillator

Because of similar structures later in quantum field theory, it is instructive to also briefly recall the harmonic oscillator in one dimension. Its Hamiltonian is given by

\[ \hat{H}(\hat{x}, \hat{p}) = \frac{1}{2}\left( \frac{\hat{p}^2}{m} + m\omega^2 \hat{x}^2 \right). \]

Employing the canonical formalism we have just set up, we easily identify the momentum operator to be \( \hat{p}(t) = m\partial_x \hat{x}(t) \), and from the Hamilton equations we find the equation of motion to be \( \partial_t^2 \hat{x} = -\omega^2 \hat{x} \), which has the well known plane wave solution \( \hat{x} \sim \exp i\omega t \).

An alternative path useful for later field theory applications is to introduce new operators, expressed in terms of the old ones,

\[ \hat{a} = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{m\omega}{\hbar}} \hat{x} + i\sqrt{\frac{1}{m\omega\hbar}} \hat{p} \right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{m\omega}{\hbar}} \hat{x} - i\sqrt{\frac{1}{m\omega\hbar}} \hat{p} \right). \]

Using the commutation relation for \( \hat{x}, \hat{p} \), one readily derives (see the preschool problems)

\[ [\hat{a}, \hat{a}^\dagger] = 1, \quad \hat{H} = -\hbar\omega \hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = \hbar \omega \hat{a}. \]
With the help of these the Hamiltonian can be rewritten in terms of the new operators:

\[ \hat{H} = \frac{1}{2} \hbar \omega \left( \hat{a}^{\dagger} \hat{a} + \hat{a} \hat{a}^{\dagger} \right) = \left( \hat{a}^{\dagger} \hat{a} + \frac{1}{2} \right) \hbar \omega. \]  

(38)

With this form of the Hamiltonian it is easy to construct a complete basis of energy eigenstates \(|n\rangle\),

\[ \hat{H} |n\rangle = E_n |n\rangle. \]  

(39)

Using the above commutation relations, one finds

\[ \hat{a}^{\dagger} \hat{H} |n\rangle = (\hat{H} \hat{a}^{\dagger} - \hbar \omega \hat{a}^{\dagger}) |n\rangle = E_n \hat{a}^{\dagger} |n\rangle, \]  

(40)

and therefore

\[ \hat{H} \hat{a}^{\dagger} |n\rangle = (E_n + \hbar \omega) \hat{a}^{\dagger} |n\rangle. \]  

(41)

Thus, the state \(\hat{a}^{\dagger} |n\rangle\) has energy \(E_n + \hbar \omega\), so that \(\hat{a}^{\dagger}\) may be regarded as a “creation operator” for a quantum with energy \(\hbar \omega\). Along the same lines one finds that \(\hat{a} |n\rangle\) has energy \(E_n - \hbar \omega\), and \(\hat{a}\) is an “annihilation operator”.

Let us introduce a vacuum state \(|0\rangle\) with no quanta excited, for which \(\hat{a} |n\rangle = 0\), because there cannot be any negative energy states. Acting with the Hamiltonian on that state we find

\[ \hat{H} |0\rangle = \hbar \omega / 2, \]  

(42)

i.e. the quantum mechanical vacuum has a non-zero energy, known as vacuum oscillation or zero point energy. Acting with a creation operator onto the vacuum state one easily finds the state with one quantum excited, and this can be repeated \(n\) times to get

\[ |1\rangle = \hat{a}^{\dagger} |0\rangle, \quad E_1 = (1 + \frac{1}{2}) \hbar \omega, \ldots \]

\[ |n\rangle = \hat{a}^{\dagger} \sqrt{n} |n - 1\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^{\dagger})^n |0\rangle, \quad E_n = (n + \frac{1}{2}) \hbar \omega. \]  

(43)

The root of the factorial is there to normalise all eigenstates to one. Finally, the number operator \(\hat{N} = \hat{a}^{\dagger} \hat{a}\) returns the number of quanta in a given energy eigenstate,

\[ \hat{N} |n\rangle = n |n\rangle. \]  

(44)

1.6 Relativistic Quantum Mechanics

So far we have only considered non-relativistic particles. In this section, we see what happens when we try to formulate a relativistic analogue of the Schrödinger equation. First, note that we can derive the non-relativistic equation starting from the energy relation

\[ E = \frac{p^2}{2m} + V(x) \]  

(45)

and replacing variables by their appropriate operators acting on a position space wavefunction \(\psi(x, t)\)

\[ E \rightarrow i \hbar \frac{\partial}{\partial t}, \quad p \rightarrow -i \hbar \nabla, \quad x \rightarrow x \]  

(46)
to give
\[
\left[-\frac{\hbar^2}{2m} \nabla^2 + V(x)\right] \psi(x, t) = i\hbar \frac{\partial \psi(x, t)}{\partial t}.
\] (47)

As we have already seen, there is a corresponding positive definite probability density
\[
\rho = |\psi(x, t)|^2 \geq 0,
\] (48)

with corresponding current
\[
j = \frac{\hbar}{2im} (\psi^* \nabla \psi - (\nabla \psi^*) \psi).
\] (49)

Can we also make a relativistic equation? By analogy with the above, we may start with the relativistic energy relation
\[
E^2 = c^2 p^2 + m^2 c^4,
\] (50)

and making the appropriate operator replacements leads to the equation
\[
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right) \phi(x, t)
\] (51)

for some wavefunction \( \phi(x, t) \). This is the Klein-Gordon equation, and one may try to form a probability density and current, as in the non-relativistic case. Firstly, one notes that to satisfy relativistic invariance, the probability density should be the zeroth component of a 4-vector \( j^\mu = (\rho, j) \) satisfying
\[
\partial_\mu j^\mu = 0.
\] (52)

In fact, one finds
\[
\rho = \frac{i\hbar}{2m} \left( \phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right),
\] (53)

with \( j \) given as before. This is not positive definite! That is, this may (and will) become negative in general, so we cannot interpret this as the probability density of a single particle.

There is another problem with the Klein-Gordon equation as it stands, that is perhaps less abstract to appreciate. The relativistic energy relation gives
\[
E = \pm \sqrt{c^2 p^2 + m^2 c^4},
\] (54)

and thus one has positive and negative energy solutions. For a free particle, one could restrict to having positive energy states only. However, an interacting particle may exchange energy with its environment, and there is nothing to stop it cascading down to energy states of more and more negative energy, thus emitting infinite amounts of energy.

We conclude that the Klein-Gordon equation does not make sense as a consistent quantum theory of a single particle. We thus need a different approach in unifying special relativity and quantum mechanics. This, as we will see, is QFT, in which we will be able to reinterpret the Klein-Gordon function as a field \( \phi(x, t) \) describing many particles.
From now on, it will be extremely convenient to work in *natural units*, in which one sets $\hbar = c = 1$. The correct factors can always be reinstated by dimensional analysis. In these units, the Klein-Gordon equation becomes

$$\left(\Box + m^2\right)\phi(x, t) = 0,$$

where

$$\Box = \partial^\mu \partial_\mu = \frac{\partial}{\partial t^2} - \nabla^2.$$  \hspace{1cm} (56)

## 2 Classical Field Theory

In the previous section, we have seen how to describe point particles, both classically and quantum mechanically. In this section, we discuss classical field theory, as a precursor to considering quantum fields. A *field* associates a mathematical object (e.g. scalar, vector, tensor, spinor...) with every point in spacetime. Examples are the temperature distribution in a room (a scalar field), or the $E$ and $B$ fields in electromagnetism (vector fields). Just as point particles can be described by Lagrangians, so can fields, although it is more natural to think in terms of *Lagrangian densities*.

### 2.1 Example: Model of an Elastic Rod

Let us consider a particular example, namely a set of point masses connected together by springs, as shown in figure 2. Assume the masses $m$ are equal, as also are the force constants of the springs $k$. Furthermore, we assume that the masses may move only longitudinally, where the $i^{th}$ displacement is $f_i$, and that the separation of adjacent masses is $\delta x$ when all $f_i$ are zero. This system is an approximation to an elastic rod, with a displacement field $f(x, t)$. To see what this field theory looks like, we may first write the total kinetic and potential energies as

$$T = \sum_i \frac{1}{2} m f_i^2, \quad V = \sum_i \frac{1}{2} k (f_{i+1} - f_i)^2$$  \hspace{1cm} (57)

respectively, where we have used Hooke’s Law for the potential energy. Thus, the Lagrangian is

$$L = T - V = \sum_i \left[ \frac{1}{2} m f_i^2 - \frac{1}{2} k (f_{i+1} - f_i)^2 \right].$$  \hspace{1cm} (58)
Clearly this system becomes a better approximation to an elastic rod as the continuum limit is approached, in which the number of masses $N \to \infty$ and the separation $\delta x \to 0$. We can then rewrite the Lagrangian as

$$ L = \sum_i \delta x \left[ \frac{1}{2} \left( \frac{m}{\delta x} \right) f_i^2 - \frac{1}{2} (k \delta x) \left( \frac{f_{i+1} - f_i}{\delta x} \right)^2 \right]. \quad (59) $$

We may recognise

$$ \lim_{\delta x \to 0} \frac{m}{\delta x} = \rho \quad (60) $$

as the density of the rod, and also define the tension

$$ \kappa = \lim_{\delta x \to 0} k \delta x. \quad (61) $$

Furthermore, the position index $i$ gets replaced by the continuous variable $x$, and one has

$$ \lim_{\delta x \to 0} \frac{f_{i+1} - f_i}{\delta x} = \frac{\partial f(x,t)}{\partial x}. \quad (62) $$

Finally, the sum over $i$ becomes an integral so that the continuum Lagrangian is

$$ L = \int dx \left[ \frac{1}{2} \rho \dot{f}(x,t)^2 - \frac{1}{2} \kappa \left( \frac{\partial f}{\partial x} \right)^2 \right]. \quad (63) $$

This is the Lagrangian for the displacement field $f(x,t)$. It depends on a function of $f$ and $\dot{f}$ which is integrated over all space coordinates (in this case there is only one, the position along the rod). We may therefore write the Lagrangian manifestly as

$$ L = \int dx \mathcal{L}[f(x,t), \dot{f}(x,t)], \quad (64) $$

where $\mathcal{L}$ is the Lagrangian density

$$ \mathcal{L}[f(x,t), \dot{f}(x,t)] = \frac{1}{2} \rho \dot{f}^2(x,t) - \frac{1}{2} \kappa \left( \frac{\partial f}{\partial x} \right)^2. \quad (65) $$

It is perhaps clear from the above example that for any field, there will always be an integration over all space dimensions, and thus it is more natural to think about the Lagrangian density rather than the Lagrangian itself. Indeed, we may construct the following dictionary between quantities in point particle mechanics, and corresponding field theory quantities (which may or may not be helpful to you in remembering the differences between particles and fields...!).

<table>
<thead>
<tr>
<th>Classical Mechanics</th>
<th>Classical Field Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(t)$</td>
<td>$\phi(x,t)$</td>
</tr>
<tr>
<td>$\dot{x}(t)$</td>
<td>$\dot{\phi}(x,t)$</td>
</tr>
<tr>
<td>Index $i$</td>
<td>Coordinate $x$</td>
</tr>
<tr>
<td>$L(x, \dot{x})$</td>
<td>$\mathcal{L}[\phi, \dot{\phi}]$</td>
</tr>
</tbody>
</table>

Note that the action for the above field theory is given, as usual, by the time integral of the Lagrangian:

$$ S = \int dt L = \int dt \int dx \mathcal{L}[f, \dot{f}]. \quad (69) $$
2.2 Relativistic Fields

In the previous section we saw how fields can be described using Lagrangian densities, and illustrated this with a non-relativistic example. Rather than derive the field equations for this case, we do this explicitly here for relativistic theories, which we will be concerned with for the rest of the course (and, indeed, the school).

In special relativity, coordinates are combined into four-vectors, \( x^\mu = (t, x_i) \) or \( x = (t, \mathbf{x}) \), whose length \( x^2 = t^2 - \mathbf{x}^2 \) is invariant under Lorentz transformations

\[
x^\mu = \Lambda^\mu_\nu x^\nu.
\]  

A general function transforms as \( f(x) \to f'(x') \), i.e. both the function and its argument transform. A Lorentz scalar is a function \( \phi(x) \) which at any given point in space-time will have the same amplitude, regardless of which inertial frame it is observed in. Consider a space-time point given by \( x \) in the unprimed frame, and \( x'(x) \) in the primed frame, where the function \( x'(x) \) can be derived from eq. (70). Observers in both the primed and unprimed frames will see the same amplitude \( \phi(x) \), although an observer in the primed frame will prefer to express this in terms of his or her own coordinate system \( x' \), hence will see

\[
\phi(x) = \phi(x(x')) = \phi'(x'),
\]  

where the latter equality defines \( \phi' \).

Equation (71) defines the transformation law for a Lorentz scalar. A vector function transforms as

\[
V'^\mu(x') = \Lambda^\mu_\nu V^\nu(x).
\]  

We will work in particular with \( \partial_\mu \phi(x) \), where \( x = x^\mu \) denotes the 4-position. Note in particular that

\[
(\partial_\mu \phi)(\partial^\mu \phi) = \left( \frac{\partial \phi}{\partial t} \right)^2 - \nabla \phi \cdot \nabla \phi
\]

\[
\partial_\mu \partial^\mu \phi = \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi.
\]

In general, a relativistically invariant scalar field theory has action

\[
S = \int d^4x L[\phi, \partial_\mu \phi],
\]  

where

\[
\int d^4x \equiv \int dt d^3x,
\]  

and \( L \) is the appropriate Lagrangian density. We can find the equations of motion satisfied by the field \( \phi(x) \) using, as in point particle mechanics, the principle of least action. The field theory form of this is that the field \( \phi(x) \) is such that the action of eq. (73) is extremised. Assuming \( \phi(x) \) is indeed such a field, we may introduce a small perturbation

\[
\phi(x) \to \phi(x) + \delta \phi(x),
\]  

17
which correspondingly perturbs the action according to

\[ S \rightarrow S + \delta S = \int d^4x \left[ L(\phi, \partial_\mu \phi) + \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right]. \tag{76} \]

Recognising the first term as the unperturbed action, one thus finds

\[ \delta S = \int d^4x \left[ \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right] \]

\[ = \left[ \frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi \right]_{\text{boundary}} + \int d^4x \left[ \frac{\partial L}{\partial \phi} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \right] \delta \phi, \]

where we have integrated by parts in the second line. Assuming the fields die away at infinity so that \( \delta \phi = 0 \) at the boundary of spacetime, the principle of least action \( \delta S = 0 \) implies

\[ \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) = \frac{\partial L}{\partial \phi}. \tag{77} \]

This is the Euler-Lagrange field equation. It tells us, given a particular Lagrangian density (which defines a particular field theory) the classical equation of motion which must be satisfied by the field \( \phi \). As a specific example, let us consider the Lagrangian density

\[ L = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2, \tag{78} \]

from which one finds

\[ \frac{\partial L}{\partial (\partial_\mu \phi)} = \partial_\mu \phi, \quad \frac{\partial L}{\partial \phi} = -m^2 \phi, \tag{79} \]

so that the Euler-Lagrange equation gives

\[ \partial_\mu \partial^\mu \phi + m^2 \phi = (\Box + m^2)\phi(x) = 0. \tag{80} \]

This is the Klein-Gordon equation! The above Lagrangian density thus corresponds to the classical field theory of a Klein-Gordon field. We see in particular that the coefficient of the quadratic term in the Lagrangian can be interpreted as the mass.

By analogy with point particle mechanics, one can define a canonical momentum field conjugate to \( \phi \):

\[ \pi(x) = \frac{\partial L}{\partial \dot{\phi}}. \tag{81} \]

Then one can define the Hamiltonian density

\[ \mathcal{H}[\phi, \pi] = \pi \dot{\phi} - L, \tag{82} \]

such that

\[ H = \int d^3x \mathcal{H}(\pi, \phi) \]

is the Hamiltonian (total energy carried by the field). For example, the Klein-Gordon field has conjugate momentum \( \pi = \dot{\phi} \), and Hamiltonian density

\[ \mathcal{H} = \frac{1}{2} \left[ \pi^2(x) + (\nabla \phi)^2 + m^2 \phi^2 \right]. \tag{84} \]
2.3 Plane wave solutions to the Klein-Gordon equation

Let us consider real solutions to Eq. (80), characterised by \( \phi^*(x) = \phi(x) \). To find them we try an ansatz of plane waves

\[
\phi(x) \propto e^{i(k^0 t - \mathbf{k} \cdot \mathbf{x})}.
\] (85)

The Klein-Gordon equation is satisfied if \((k^0)^2 - \mathbf{k}^2 = m^2\) so that

\[
k^0 = \pm \sqrt{k^2 + m^2}.
\] (86)

Defining the energy as

\[
E(k) = \sqrt{k^2 + m^2} > 0,
\] (87)

we obtain two types of solution which read

\[
\phi_+(x) \propto e^{iE(k) t - \mathbf{k} \cdot \mathbf{x}}, \quad \phi_-(x) \propto e^{-iE(k) t - \mathbf{k} \cdot \mathbf{x}}.
\] (88)

We may interpret these as positive and negative energy solutions, such that it does not matter which branch of the square root we take in eq. (87) (it is conventional, however, to define energy as a positive quantity). The general solution is a superposition of \(\phi_+\) and \(\phi_-\). Using

\[
E(k) t - \mathbf{k} \cdot \mathbf{x} = k^\mu k_\mu = \mu k^\mu = k \cdot x
\] (89)

this solution reads

\[
\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E(k)} \left( e^{ik \cdot x} \alpha^*(k) + e^{-ik \cdot x} \alpha(k) \right),
\] (90)

where \(\alpha(k)\) is an arbitrary complex coefficient. Note that the coefficients of the positive and negative exponentials are related by complex conjugation. This ensures that the field \(\phi(x)\) is real (as can be easily verified from eq. (90)), consistent with the Lagrangian we wrote down. Such a field has applications in e.g. the description of neutral mesons. We can also write down a Klein-Gordon Lagrangian for a complex field \(\phi\). This is really two independent fields (i.e. \(\phi\) and \(\phi^*\)), and thus can be used to describe a system of two particles (e.g. charged meson pairs). To simplify the discussion in this course, we will explicitly consider the real Klein-Gordon field. Note that the factors of 2 and \(\pi\) in eq. (90) are conventional, and the inverse power of the energy is such that the measure of integration is Lorentz invariant (problem 2.1), so that the whole solution is written in a manifestly Lorentz invariant way.

2.4 Symmetries and Conservation Laws

As was the case in point particle mechanics, one may relate symmetries of the Lagrangian density to conserved quantities in field theory. For example, consider the invariance of \(\mathcal{L}\) under space-time translations

\[
x^\mu \rightarrow x^\mu + \epsilon^\mu,
\] (91)

where \(\epsilon^\mu\) is constant. Under such a transformation one has

\[
\mathcal{L}(x^\mu + \epsilon^\mu) = \mathcal{L}(x^\mu) + \epsilon^\mu \partial_\mu \mathcal{L}(x^\mu) + \ldots
\] (92)

\[
\phi(x^\mu + \epsilon^\mu) = \phi(x^\mu) + \epsilon^\mu \partial_\mu \phi(x^\mu) + \ldots
\] (93)

\[
\partial_\nu \phi(x^\mu + \epsilon^\mu) = \partial_\nu \phi(x^\mu) + \epsilon^\mu \partial_\mu \partial_\nu \phi(x^\mu) + \ldots,
\] (94)
where we have used Taylor’s theorem. But if $\mathcal{L}$ does not explicitly depend on $x^\mu$ (i.e. only through $\phi$ and $\partial_\mu \phi$) then one has

$$\mathcal{L}(x^\mu + \epsilon^\mu) = \mathcal{L}[\phi(x^\mu + \epsilon^\mu), \partial_\nu \phi(x^\mu + \epsilon^\mu)]$$

$$= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \delta (\partial_\nu \phi) + \ldots$$

$$= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \epsilon^\mu \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \epsilon^\mu \partial_\mu \partial_\nu \phi + \ldots,$$

where we have used the fact that $\delta \phi = \epsilon^\mu \partial_\mu \phi$ in the third line, and all functions on the right-hand side are evaluated at $x^\mu$. One may replace $\partial \mathcal{L}/\partial \phi$ by the LHS of the Euler-Lagrange equation to get

$$\mathcal{L}(x^\mu + \epsilon^\mu) = \mathcal{L} + \partial_\nu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\mu \phi \right] \epsilon^\mu,$$

and equating this with the alternative expression above, one finds

$$\partial_\nu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\mu \phi \right] \epsilon^\mu = \epsilon^\mu \partial_\mu \mathcal{L}.$$  

If this is true for all $\epsilon^\mu$, then one has

$$\partial^\nu \Theta_{\nu \mu} = 0,$$

where

$$\Theta_{\nu \mu} = \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\mu \phi - g_{\mu \nu} \mathcal{L}$$

is the energy-momentum tensor. We can see how this name arises by considering the components explicitly, for the case of the Klein Gordon field. One then finds

$$\Theta_{00} = \frac{\partial \mathcal{L}}{\partial \phi} \dot{\phi} - g_{00} \mathcal{L} = \pi \dot{\phi} - \mathcal{L} = \mathcal{H},$$

$$\Theta_{0j} = \frac{\partial \mathcal{L}}{\partial \phi} \partial_j \phi - g_{0j} \mathcal{L} = \pi \partial_j \phi \quad (j = 1 \ldots 3).$$

One then sees that $\Theta_{00}$ is the energy density carried by the field. Its conservation can then be shown by considering

$$\frac{\partial}{\partial t} \int_V d^3 x \Theta_{00} = \int_V d^3 x \partial_0 \Theta_{00}$$

$$= \int_V d^3 x \partial^j \Theta_{j0} = \int_S dS_j \cdot \Theta_{0j} = 0,$$

where we have used Eq. (100) in the second line. The Hamiltonian density is a conserved quantity, provided that there is no energy flow through the surface $S$ which encloses the volume $V$. In a
similar manner one can show that the 3-momentum $p_j$, which is related to $\Theta_{0j}$, is conserved as well. It is then useful to define a conserved energy-momentum four-vector

$$P_\mu = \int d^3x \, \Theta_{0\mu}.$$  \hspace{1cm} (105)

In analogy to point particle mechanics, we thus see that invariances of the Lagrangian density correspond to conservation laws. An entirely analogous procedure leads to conserved quantities like angular momentum and spin. Furthermore one can study so-called internal symmetries, i.e. ones which are not related to coordinate but other transformations. Examples are conservation of all kinds of charges, isospin, etc.

We have thus established the Lagrange-Hamilton formalism for classical field theory: we derived the equation of motion (Euler-Lagrange equation) from the Lagrangian and introduced the conjugate momentum. We then defined the Hamiltonian (density) and considered conservation laws by studying the energy-momentum tensor $\Theta_{\mu\nu}$.

### 3 Quantum Field Theory: Free Fields

#### 3.1 Canonical Field Quantisation

In the previous sections we have reviewed the classical and quantum mechanics of point particles, and also classical field theory. We used the canonical quantisation procedure in discussing quantum mechanics, whereby classical variables are replaced by operators, which have non-trivial commutation relations. In this section, we see how to apply this procedure to fields, taking the explicit example of the Klein-Gordon field discussed previously. This is, as yet, a non-interacting field theory, and we will discuss how to deal with interactions later on in the course.

The Klein-Gordon Lagrangian density has the form

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2.$$  \hspace{1cm} (106)

We have seen that in field theory the field $\phi(x)$ plays the role of the coordinates in ordinary point particle mechanics, and we defined a canonically conjugate momentum, $\pi(x) = \partial \mathcal{L}/\partial \dot{\phi} = \dot{\phi}(x)$. We then continue the analogy to point mechanics through the quantisation procedure, i.e. we now take our canonical variables to be operators,

$$\phi(x) \to \hat{\phi}(x), \quad \pi(x) \to \hat{\pi}(x).$$  \hspace{1cm} (107)

Next we impose equal-time commutation relations on them,

$$\left[ \hat{\phi}(x, t), \hat{\pi}(y, t) \right] = i\delta^3(x - y),$$  \hspace{1cm} (108)

$$\left[ \hat{\phi}(x, t), \hat{\phi}(y, t) \right] = \left[ \hat{\pi}(x, t), \hat{\pi}(y, t) \right] = 0.$$  \hspace{1cm} (109)

As in the case of quantum mechanics, the canonical variables commute among themselves, but not the canonical coordinate and momentum with each other. Note that the commutation relation is entirely analogous to the quantum mechanical case. There would be an $\hbar$, if it hadn’t been set to
one earlier, and the delta-function accounts for the fact that we are dealing with fields. It is zero
if the fields are evaluated at different space-time points.

After quantisation, our fields have turned into field operators. Note that within the relativistic
formulation they depend on time, and hence they are Heisenberg operators.

In the previous paragraph we have formulated commutation relations for fields evaluated at equal
time, which is clearly a special case when considering fields at general \( x, y \). The reason has to do
with maintaining causality in a relativistic theory. Let us recall the light cone about an event at \( y \),
as in Fig. 3. One important postulate of special relativity states that no signal and no interaction
can travel faster than the speed of light. This has important consequences about the way in which
different events can affect each other. For instance, two events which are characterised by space-
time points \( x^\mu \) and \( y^\mu \) are said to be causal if the distance \((x - y)^2\) is time-like, i.e. \((x - y)^2 > 0\).

By contrast, two events characterised by a space-like separation, i.e. \((x - y)^2 < 0\), cannot affect
each other, since the point \( x \) is not contained inside the light cone about \( y \).

In non-relativistic Quantum Mechanics the commutation relations among operators indicate whether
precise and independent measurements of the corresponding observables can be made. If the com-
mmutator does not vanish, then a measurement of one observable affects that of the other. From
the above it is then clear that the issue of causality must be incorporated into the commutation
relations of the relativistic version of our quantum theory: whether or not independent and precise
measurements of two observables can be made depends also on the separation of the 4-vectors char-
acterising the points at which these measurements occur. Clearly, events with space-like separations
cannot affect each other, and hence all fields must commute,

\[
\left[ \hat{\phi}(x), \hat{\phi}(y) \right] = \left[ \hat{\pi}(x), \hat{\pi}(y) \right] = \left[ \hat{\phi}(x), \hat{\pi}(y) \right] = 0 \quad \text{for} \quad (x - y)^2 < 0.
\]  

This condition is sometimes called micro-causality. Writing out the four-components of the time
interval, we see that as long as \(|t' - t| < |x - y|\), the commutator vanishes in a finite interval \(|t' - t|\).

It also vanishes for \( t' = t \), as long as \( x \neq y \). Only if the fields are evaluated at an equal space-time
point can they affect each other, which leads to the equal-time commutation relations above. They
can also affect each other everywhere within the light cone, i.e. for time-like intervals. It is not

---

**Figure 3:** The light cone about \( y \). Events occurring at points \( x \) and \( y \) are said to be time-like
(space-like) if \( x \) is inside (outside) the light cone about \( y \).
hard to show that in this case (e.g. problem 3.1)

\[
\left[ \hat{\phi}(x), \hat{\phi}(y) \right] = [\hat{\pi}(x), \hat{\pi}(y)] = 0, \quad \text{for} \quad (x - y)^2 > 0 \quad (111)
\]

\[
\left[ \hat{\phi}(x), \hat{\pi}(y) \right] = \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left( e^{ip(x-y)} + e^{-ip(x-y)} \right) . \quad (112)
\]

n.b. since the 4-vector dot product \( p \cdot (x - y) \) depends on \( p_0 = \sqrt{p^2 + m^2} \), one cannot trivially carry out the integrals over \( d^3p \) here.

### 3.2 Creation and annihilation operators

After quantisation, the Klein-Gordon equation we derived earlier turns into an equation for operators. For its solution we simply promote the classical plane wave solution, Eq. (90), to operator status,

\[
\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E(k)} \left( e^{ik \cdot x} \hat{a}(k) + e^{-ik \cdot x} \hat{a}^\dagger(k) \right) . \quad (113)
\]

Note that the complex conjugation of the Fourier coefficient turned into hermitian conjugation for an operator.

Let us now solve for the operator coefficients of the positive and negative energy solutions. In order to do so, we invert the Fourier integrals for the field and its time derivative,

\[
\int d^3x \hat{\phi}(x, t)e^{ikx} = \frac{1}{2E} \left[ \hat{a}(k) + \hat{a}^\dagger(k) e^{2ik_0x_0} \right], \quad (114)
\]

\[
\int d^3x \hat{\dot{\phi}}(x, t)e^{ikx} = -\frac{i}{2} \left[ \hat{a}(k) - \hat{a}^\dagger(k) e^{2ik_0x_0} \right], \quad (115)
\]

and then build the linear combination \( iE(k)(114)-(115) \) to find

\[
\int d^3x \left[ iE(k)\hat{\phi}(x,t) - \hat{\dot{\phi}}(x,t) \right] e^{ikx} = i\hat{a}(k), \quad (116)
\]

Following a similar procedure for \( \hat{a}^\dagger(k) \), and using \( \hat{\pi}(x) = \hat{\dot{\phi}}(x) \) we find

\[
\hat{a}(k) = \int d^3x \left[ E(k)\hat{\phi}(x,t) + i\hat{\pi}(x,t) \right] e^{ikx}, \quad (117)
\]

\[
\hat{a}^\dagger(k) = \int d^3x \left[ E(k)\hat{\phi}(x,t) - i\hat{\pi}(x,t) \right] e^{-ikx}. \quad (118)
\]

Note that, as Fourier coefficients, these operators do not depend on time, even though the right hand side does contain time variables. Having expressions in terms of the canonical field variables \( \hat{\phi}(x), \hat{\pi}(x) \), we can now evaluate the commutators for the Fourier coefficients. Expanding everything out and using the commutation relations Eq. (109), we find

\[
[\hat{a}^\dagger(k_1), \hat{a}(k_2)] = 0 \quad (119)
\]

\[
[\hat{a}(k_1), \hat{a}(k_2)]= 0 \quad (120)
\]

\[
[\hat{a}(k_1), \hat{a}^\dagger(k_2)] = (2\pi)^3 2E(k_1)\delta^3(k_1 - k_2) \quad (121)
\]
We easily recognise these for every $k$ to correspond to the commutation relations for the harmonic oscillator, Eq. (37). This motivates us to also express the Hamiltonian and the energy momentum four-vector of our quantum field theory in terms of these operators. To do this, first note that the Hamiltonian is given by the integral of the Hamiltonian density (eq. (84)) over all space. One may then substitute eq. (113) to yield (see the problem sheet)

$$\hat{H} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2E(k)} \ E(k) \left( \hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k) \right),$$

$$\hat{P} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2E(k)} \ k \left( \hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k) \right).$$

We thus find that the Hamiltonian and the momentum operator are nothing but a continuous sum of excitation energies/momenta of one-dimensional harmonic oscillators! After a minute of thought this is not so surprising. We expanded the solution of the Klein-Gordon equation into a superposition of plane waves with momenta $k$. But of course a plane wave solution with energy $E(k)$ is also the solution to a one-dimensional harmonic oscillator with the same energy. Hence, our free scalar field is simply a collection of infinitely many harmonic oscillators distributed over the whole energy/momentum range. These energies sum up to that of the entire system. We have thus reduced the problem of handling our field theory to oscillator algebra. From the harmonic oscillator we know already how to construct a complete basis of energy eigenstates, and thanks to the analogy of the previous section we can take this over to our quantum field theory.

### 3.3 Energy of the vacuum state and renormalisation

In complete analogy we begin again with the postulate of a vacuum state $|0\rangle$ with norm one, which is annihilated by the action of the operator $a$,

$$\langle 0|0 \rangle = 1, \quad \hat{a}(k)|0\rangle = 0 \quad \text{for all} \quad k.$$  

(124)

Let us next evaluate the energy of this vacuum state, by taking the expectation value of the Hamiltonian,

$$E_0 = \langle 0|\hat{H}|0 \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2E(k)} E(k) \left\{ \langle 0|\hat{a}^\dagger(k)\hat{a}(k)|0 \rangle + \langle 0|\hat{a}(k)\hat{a}^\dagger(k)|0 \rangle \right\}.$$  

(125)

The first term in curly brackets vanishes, since $a$ annihilates the vacuum. The second can be rewritten as

$$\hat{a}(k)\hat{a}^\dagger(k)|0\rangle = \left\{ \left[ \hat{a}(k), \hat{a}^\dagger(k) \right] + \hat{a}^\dagger(k)\hat{a}(k) \right\} |0\rangle.$$  

(126)

It is now the second term which vanishes, whereas the first can be replaced by the value of the commutator. Thus we obtain

$$E_0 = \langle 0|\hat{H}|0 \rangle = \delta^3(0) \frac{1}{2} \int d^3k E(k) = \delta^3(0) \frac{1}{2} \int d^3k \sqrt{k^2 + m^2} = \infty,$$  

(127)

which means that the energy of the ground state is infinite! This result seems rather paradoxical, but it can be understood again in terms of the harmonic oscillator. Recall that the simple quantum mechanical oscillator has a finite zero-point energy. As we have seen above, our field theory corresponds to an infinite collection of harmonic oscillators, i.e. the vacuum receives an infinite number of zero point contributions, and its energy thus diverges.
This is the first of frequent occurrences of infinities in quantum field theory. Fortunately, it is not too hard to work around this particular one. Firstly, we note that nowhere in nature can we observe absolute values of energy, all we can measure are energy differences relative to some reference scale, at best the one of the vacuum state, \(|0\rangle\). In this case it does not really matter what the energy of the vacuum is. This then allows us to redefine the energy scale, by always subtracting the (infinite) vacuum energy from any energy we compute. This process is called “renormalisation”.

We then define the renormalised vacuum energy to be zero, and take it to be the expectation value of a renormalised Hamiltonian,

\[
E_0^R \equiv \langle 0 | \hat{H}^R | 0 \rangle = 0.
\] (128)

According to this recipe, the renormalised Hamiltonian is our original one, minus the (unrenormalised) vacuum energy,

\[
\hat{H}^R = \hat{H} - E_0
\] (129)

\[
= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2E(k)} E(k) \left\{ \hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k) - \langle 0 | \hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k) | 0 \rangle \right\}
\]

\[
= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2E(k)} E(k) \left\{ 2\hat{a}^\dagger(k)\hat{a}(k) + \left[ \hat{a}(k), \hat{a}^\dagger(k) \right] - \langle 0 | \left[ \hat{a}(k), \hat{a}^\dagger(k) \right] | 0 \rangle \right\}. \] (130)

Here the subtraction of the vacuum energy is shown explicitly, and we can rewrite it as

\[
\hat{H}^R = \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \hat{a}^\dagger(p)\hat{a}(p)
\]

\[
+ \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \left\{ \left[ \hat{a}(p), \hat{a}^\dagger(p) \right] - \langle 0 | \left[ \hat{a}(p), \hat{a}^\dagger(p) \right] | 0 \rangle \right\}. \] (131)

The operator \(\hat{H}^{\text{vac}}\) ensures that the vacuum energy is properly subtracted: if \(|\psi\rangle\) and \(|\psi'\rangle\) denote arbitrary \(N\)-particle states, then one can convince oneself that \(\langle \psi' | \hat{H}^{\text{vac}} | \psi \rangle = 0\). In particular we now find that

\[
\langle 0 | \hat{H}^R | 0 \rangle = 0, \] (132)

as we wanted. A simple way to automatise the removal of the vacuum contribution is to introduce normal ordering. Normal ordering means that all annihilation operators appear to the right of any creation operator. The notation is

\[
: \hat{a}\hat{a}^\dagger : = \hat{a}^\dagger\hat{a}, \] (133)

i.e. the normal-ordered operators are enclosed within colons. For instance

\[
:\frac{1}{2} \left( \hat{a}^\dagger(p)\hat{a}(p) + \hat{a}(p)\hat{a}^\dagger(p) \right) : = \hat{a}^\dagger(p)\hat{a}(p). \] (134)

It is important to keep in mind that \(\hat{a}\) and \(\hat{a}^\dagger\) always commute inside \(\cdots :\). This is true for an arbitrary string of \(\hat{a}\) and \(\hat{a}^\dagger\). With this definition we can write the normal-ordered Hamiltonian as

\[
\hat{H} : = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \left( \hat{a}^\dagger(p)\hat{a}(p) + \hat{a}(p)\hat{a}^\dagger(p) \right):
\]

\[
= \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \hat{a}^\dagger(p)\hat{a}(p), \] (135)
and thus have the relation
\[ \hat{H}^R = \hat{H} + \hat{H}^{\text{vac}}. \] (136)

Hence, we find that
\[ \langle \psi' | \hat{H} | \psi \rangle = \langle \psi' | \hat{H}^R | \psi \rangle, \] (137)
and, in particular, \( \langle 0 | \hat{H} | 0 \rangle = 0 \). The normal ordered Hamiltonian thus produces a renormalised, sensible result for the vacuum energy.

### 3.4 Fock space and Particles

After this lengthy grappling with the vacuum state, we can continue to construct our basis of states in analogy to the harmonic oscillator, making use of the commutation relations for the operators \( \hat{a}, \hat{a}^\dagger \). In particular, we define the state \( |k\rangle \) to be the one obtained by acting with the operator \( a^\dagger(k) \) on the vacuum,
\[ |k\rangle = \hat{a}^\dagger(k)|0\rangle. \] (138)

Using the commutator, its norm is found to be
\[ \langle k|k'\rangle = \langle 0|\hat{a}(k)\hat{a}^\dagger(k')|0\rangle = \langle 0|\hat{a}(k), \hat{a}^\dagger(k')|0\rangle + \langle 0|\hat{a}^\dagger(k')\hat{a}(k)|0\rangle \] (139)
\[ = (2\pi)^32E(k)\delta^3(k-k'), \] (140)

since the last term in the first line vanishes (\( \hat{a}(k) \) acting on the vacuum). Next we compute the energy of this state, making use of the normal ordered Hamiltonian,
\[ : \hat{H} : |k\rangle = \int \frac{d^3k'}{(2\pi)^32E(k')}E(k')\hat{a}^\dagger(k')\hat{a}(k'|0\rangle \] (141)
\[ = \int \frac{d^3k'}{(2\pi)^32E(k')}E(k')(2\pi)^32E(k)\delta(k-k')\hat{a}^\dagger(k)|0\rangle \] (142)
\[ = E(k)\hat{a}^\dagger(k)|0\rangle = E(k)|k\rangle, \] (143)

and similarly one finds
\[ : \hat{\mathbf{P}} : |k\rangle = k|k\rangle. \] (144)

Observing that the normal ordering did its job and we obtain renormalised, finite results, we may now interpret the state \( |k\rangle \). It is a one-particle state for a relativistic particle of mass \( m \) and momentum \( \mathbf{k} \), since acting on it with the energy-momentum operator returns the relativistic one particle energy-momentum dispersion relation, \( E(k) = \sqrt{k^2 + m^2} \). The \( a^\dagger(k) \), \( a(k) \) are creation and annihilation operators for particles of momentum \( \mathbf{k} \).

In analogy to the harmonic oscillator, the procedure can be continued to higher states. One easily checks that (problem 3.4)
\[ : \hat{P}_\mu : \hat{a}^\dagger(k_2)\hat{a}^\dagger(k_1)|0\rangle = (k_1^\mu + k_2^\mu)\hat{a}^\dagger(k_2)\hat{a}^\dagger(k_1)|0\rangle, \] (145)

and so the state
\[ |k_2, k_1\rangle = \frac{1}{\sqrt{2!}}\hat{a}^\dagger(k_2)\hat{a}^\dagger(k_1)|0\rangle \] (146)
is a two-particle state (the factorial is there to have it normalised in the same way as the one-particle state), and so on for higher states. These are called Fock states in the textbooks (formally
speaking, a Fock space is a tensor product of Hilbert spaces, where the latter occur in ordinary Quantum Mechanics).

At long last we can now see how the field in our free quantum field theory is related to particles. A particle of momentum \( \mathbf{k} \) corresponds to an excited Fourier mode of a field. Since the field is a superposition of all possible Fourier modes, one field is enough to describe all possible configurations representing one or many particles of the same kind in any desired momentum state. 

There are some rather profound ideas here about how nature works at fundamental scales. In classical physics we have matter particles, and forces which act on those particles. These forces can be represented by fields, such that fields and particles are distinct concepts. In non-relativistic quantum mechanics, one unifies the concept of waves and particles (particles can have wave-like characteristics), but fields are still distinct (e.g. one may quantise a particle in an electromagnetic field in QM, provided the latter is treated classically). Taking into account the effects of relativity for both particles and fields, one finds in QFT that all particles are excitation quanta of fields. That is, the concepts of field and particle are no longer distinct, but actually manifestations of the same thing, namely quantum fields. In this sense, QFT is more fundamental than either of its preceding theories. Each force field and each matter field have particles associated with it.

Returning to our theory for the free Klein-Gordon field, let us investigate what happens under interchange of the two particles. Since \( [\hat{a}^\dagger(\mathbf{k}_1), \hat{a}^\dagger(\mathbf{k}_2)] = 0 \) for all \( \mathbf{k}_1, \mathbf{k}_2 \), we see that
\[
|k_2, k_1\rangle = |k_1, k_2\rangle, \tag{147}
\]
and hence the state is symmetric under interchange of the two particles. Thus, the particles described by the scalar field are bosons.

Finally we complete the analogy to the harmonic oscillator by introducing a number operator
\[
\hat{N}(\mathbf{k}) = \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}), \quad \hat{N} = \int d^3k \; \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}), \tag{148}
\]
which gives us the number of bosons described by a particular Fock state,
\[
\hat{N}|0\rangle = 0, \quad \hat{N}|\mathbf{k}\rangle = |\mathbf{k}\rangle, \quad \hat{N}|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle = n|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle. \tag{149}
\]

Of course the normal-ordered Hamiltonian can now simply be given in terms of this operator,
\[
: \hat{H} : = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} E(\mathbf{k})\hat{N}(\mathbf{k}), \tag{150}
\]
i.e. when acting on a Fock state it simply sums up the energies of the individual particles to give
\[
: \hat{H} : |\mathbf{k}_1 \ldots \mathbf{k}_n\rangle = (E(\mathbf{k}_1) + \ldots + E(\mathbf{k}_n)) |\mathbf{k}_1 \ldots \mathbf{k}_n\rangle. \tag{151}
\]

This concludes the quantisation of our free scalar field theory. We have followed the canonical quantisation procedure familiar from quantum mechanics. Due to the infinite number of degrees of freedom, we encountered a divergent vacuum energy, which we had to renormalise. The renormalised Hamiltonian and the Fock states that we constructed describe free relativistic, uncharged spin zero particles of mass \( m \), such as neutral pions, for example.

If we want to describe charged pions as well, we need to introduce complex scalar fields, the real and imaginary parts being necessary to describe opposite charges. For particles with spin we need
still more degrees of freedom and use vector or spinor fields, which have the appropriate rotation and Lorentz transformation properties. For fermion fields (which satisfy the Dirac equation rather than the Klein-Gordon equation), one finds that the condition of a positive-definite energy density requires that one impose anti-commutation relations rather than commutation relations. This in turn implies that multiparticle states are antisymmetric under interchange of identical fermions, which we recognise as the Pauli exclusion principle. Thus, not only does QFT provide a consistent theory of relativistic multiparticle systems; it also allows us to “derive” the Pauli principle, which is put in by hand in non-relativistic quantum mechanics.

More details on vector and spinor fields can be found in the other courses at this school. Here, we continue to restrict our attention to scalar fields, so as to more clearly illustrate what happens when interactions are present.

4 Quantum Field Theory: Interacting Fields

So far we have seen how to quantise the Klein-Gordon Lagrangian, and seen that this describes free scalar particles. For interesting physics, however, we need to know how to describe interactions, which lead to nontrivial scattering processes. This is the subject of this section.

From now on we shall always discuss quantised real scalar fields. It is then convenient to drop the “hats” on the operators that we have considered up to now. Interactions can be described by adding a term $L_{\text{int}}$ to the Lagrangian density, so that the full result $L$ is given by

$$L = L_0 + L_{\text{int}}$$

where

$$L_0 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

is the free Lagrangian density discussed before. The Hamiltonian density of the interaction is related to $L_{\text{int}}$ simply by

$$H_{\text{int}} = H - H_0,$$

where $H_0$ is the free Hamiltonian. If the interaction Lagrangian only depends on $\phi$ (we will consider such a case later in the course), one has

$$H_{\text{int}} = -L_{\text{int}},$$

as can be easily shown from the definition above. We shall leave the details of $L_{\text{int}}$ unspecified for the moment. What we will be concerned with mostly are scattering processes, in which two initial particles with momenta $p_1$ and $p_2$ scatter, thereby producing a number of particles in the final state, characterised by momenta $k_1, \ldots, k_n$. This is schematically shown in Fig. 4. Our task is to find a description of such a scattering process in terms of the underlying quantum field theory.

4.1 The $S$-matrix

The timescales over which interactions happen are extremely short. The scattering (interaction) process takes place during a short interval around some particular time $t$ with $-\infty \ll t \ll \infty$. 
Long before $t$, the incoming particles evolve independently and freely. They are described by a field operator $\phi_{\text{in}}$ defined through
\[
\lim_{t \to -\infty} \phi(x) = \phi_{\text{in}}(x),
\]
which acts on a corresponding basis of $|\text{in}\rangle$ states. Long after the collision the particles in the final state evolve again like in the free theory, and the corresponding operator is
\[
\lim_{t \to +\infty} \phi(x) = \phi_{\text{out}}(x),
\]
acting on states $|\text{out}\rangle$. The fields $\phi_{\text{in}}$, $\phi_{\text{out}}$ are the asymptotic limits of the Heisenberg operator $\phi$. They both satisfy the free Klein-Gordon equation, i.e.
\[
(\Box + m^2)\phi_{\text{in}}(x) = 0, \quad (\Box + m^2)\phi_{\text{out}}(x) = 0.
\]
Operators describing free fields can be expressed as a superposition of plane waves (see Eq. (113)). Thus, for $\phi_{\text{in}}$ we have
\[
\phi_{\text{in}}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E(k)} \left( e^{ik \cdot x} a_{\text{in}}^\dagger(k) + e^{-ik \cdot x} a_{\text{in}}(k) \right),
\]
with an entirely analogous expression for $\phi_{\text{out}}(x)$. Note that the operators $a^\dagger$ and $a$ also carry subscripts “in” and “out”.

The above discussion assumes that the interaction is such that we can talk about free particles at asymptotic times $t \to \pm \infty$ i.e. that the interaction is only present at intermediate times. This is not always a reasonable assumption e.g. it does not encompass the phenomenon of bound states, in which incident particles form a composite object at late times, which no longer consists of free particles. Nevertheless, the assumption will indeed allow us to discuss scattering processes, which is the aim of this course. Note that we can only talk about well-defined particle states at $t \to \pm \infty$ (the states labelled by “in” and “out” above), as only at these times do we have a free theory, and thus know what the spectrum of states is (using the methods of section 3). At general times $t$, the interaction is present, and it is not possible in general to solve for the states of the quantum field theory. Remarkably, we will end up seeing that we can ignore all the complicated stuff at intermediate times, and solve for scattering probabilities purely using the properties of the asymptotic fields.
At the asymptotic times \( t = \pm \infty \), we can use the creation operators \( a^\dagger_{\text{in}} \) and \( a^\dagger_{\text{out}} \) to build up Fock states from the vacuum. For instance

\[
\begin{align*}
a^\dagger_{\text{in}}(p_1) a^\dagger_{\text{in}}(p_2)|0\rangle &= |p_1, p_2; \text{in}\rangle, \\
a^\dagger_{\text{out}}(k_1) \cdots a^\dagger_{\text{out}}(k_n)|0\rangle &= |k_1, \ldots, k_n; \text{out}\rangle.
\end{align*}
\]

We must now distinguish between Fock states generated by \( a^\dagger_{\text{in}} \) and \( a^\dagger_{\text{out}} \), and therefore we have labelled the Fock states accordingly. In eqs. (160) and (161) we have assumed that there is a stable and unique vacuum state of the free theory (the vacuum at general times \( t \) will be that of the full interacting theory, and thus differ from this in general):

\[
|0\rangle = |0; \text{in}\rangle = |0; \text{out}\rangle.
\]  

Mathematically speaking, the \( a^\dagger_{\text{in}} \)'s and \( a^\dagger_{\text{out}} \)'s generate two different bases of the Fock space. Since the physics that we want to describe must be independent of the choice of basis, expectation values expressed in terms of “in” and “out” operators and states must satisfy

\[
\langle \text{in} | \phi_{\text{in}}(x) | \text{in} \rangle = \langle \text{out} | \phi_{\text{out}}(x) | \text{out} \rangle.
\]

Here |\text{in}\rangle and |\text{out}\rangle denote generic “in” and “out” states. We can relate the two bases by introducing a unitary operator \( S \) such that

\[
\begin{align*}
\phi_{\text{in}}(x) &= S \phi_{\text{out}}(x) S^\dagger, \\
|\text{in}\rangle &= S |\text{out}\rangle, \quad |\text{out}\rangle = S^\dagger |\text{in}\rangle, \quad S^\dagger S = 1.
\end{align*}
\]

\( S \) is called the **S-matrix** or **S-operator**. Note that the plane wave solutions of \( \phi_{\text{in}} \) and \( \phi_{\text{out}} \) also imply that

\[
a^\dagger_{\text{in}} = S a^\dagger_{\text{out}} S^\dagger, \quad a_{\text{in}} = S a_{\text{out}} S^\dagger.
\]

By comparing “in” with “out” states one can extract information about the interaction – this is the very essence of detector experiments, where one tries to infer the nature of the interaction by studying the products of the scattering of particles that have been collided with known energies. As we will see below, this information is contained in the elements of the S-matrix.

By contrast, in the absence of any interaction, i.e. for \( \mathcal{L}_{\text{int}} = 0 \) the distinction between \( \phi_{\text{in}} \) and \( \phi_{\text{out}} \) is not necessary. They can thus be identified, and then the relation between different bases of the Fock space becomes trivial, \( S = 1 \), as one would expect.

What we are ultimately interested in are transition amplitudes between an initial state \( i \) of, say, two particles of momenta \( p_1, p_2 \), and a final state \( f \), for instance \( n \) particles of unequal momenta. The transition amplitude is then given by

\[
\langle f, \text{out} | i, \text{in} \rangle = \langle f, \text{out} | S | i, \text{out} \rangle = \langle f, \text{in} | S | i, \text{in} \rangle \equiv S_{fi}.
\]

The S-matrix element \( S_{fi} \) therefore describes the transition amplitude for the scattering process in question. The scattering cross section, which is a measurable quantity, is then proportional to \( |S_{fi}|^2 \). All information about the scattering is thus encoded in the S-matrix, which must therefore be closely related to the interaction Hamiltonian density \( \mathcal{H}_{\text{int}} \). However, before we try to derive the relation between \( S \) and \( \mathcal{H}_{\text{int}} \) we have to take a slight detour.
4.2 More on time evolution: Dirac picture

The operators \( \phi(x,t) \) and \( \pi(x,t) \) which we have encountered are Heisenberg fields and thus time-dependent. The state vectors are time-independent in the sense that they do not satisfy a non-trivial equation of motion. Nevertheless, state vectors in the Heisenberg picture can carry a time label. For instance, the “in”-states of the previous subsection are defined at \( t = -\infty \). The relation of the Heisenberg operator \( \phi_H(x) \) with its counterpart \( \phi_S \) in the Schrödinger picture is given by

\[
\phi_H(x,t) = e^{iHt} \phi_S e^{-iHt}, \quad H = H_0 + H_{\text{int}},
\]

Note that this relation involves the full Hamiltonian \( H = H_0 + H_{\text{int}} \) in the interacting theory. We have so far found solutions to the Klein-Gordon equation in the free theory, and so we know how to handle time evolution in this case. However, in the interacting case the Klein-Gordon equation has an extra term,

\[
(\Box + m^2)\phi(x) + \frac{\partial V_{\text{int}}(\phi)}{\partial \phi} = 0,
\]

due to the potential of the interactions. Apart from very special cases of this potential, the equation cannot be solved anymore in closed form, and thus we no longer know the time evolution. It is therefore useful to introduce a new quantum picture for the interacting theory, in which the time dependence is governed by \( H_0 \) only. This is the so-called Dirac or Interaction picture. The relation between fields in the Interaction picture, \( \phi_I \), and in the Schrödinger picture, \( \phi_S \), is given by

\[
\phi_I(x,t) = e^{iH_0 t} \phi_S e^{-iH_0 t}.
\]

At \( t = -\infty \) the interaction vanishes, i.e. \( H_{\text{int}} = 0 \), and hence the fields in the Interaction and Heisenberg pictures are identical, i.e. \( \phi_H(x,t) = \phi_I(x,t) \) for \( t \to -\infty \). The relation between \( \phi_H \) and \( \phi_I \) can be worked out easily:

\[
\phi_H(x,t) = e^{iHt} \phi_S e^{-iHt} = e^{iHt} e^{-iH_0 t} e^{iH_0 t} \phi_S e^{-iH_0 t} e^{iH_0 t} = U^{-1}(t) \phi_I(x,t) U(t),
\]

where we have introduced the unitary operator \( U(t) \)

\[
U(t) = e^{iH_0 t} e^{-iHt}, \quad U^\dagger U = 1.
\]

The field \( \phi_H(x,t) \) contains the information about the interaction, since it evolves over time with the full Hamiltonian. In order to describe the “in” and “out” field operators, we can now make the following identifications:

\[
t \to -\infty : \quad \phi_{\text{in}}(x,t) = \phi_I(x,t) = \phi_H(x,t),
\]

\[
t \to +\infty : \quad \phi_{\text{out}}(x,t) = \phi_H(x,t).
\]

Furthermore, since the fields \( \phi_I \) evolve over time with the free Hamiltonian \( H_0 \), they always act in the basis of “in” vectors, such that

\[
\phi_{\text{in}}(x,t) = \phi_I(x,t), \quad -\infty < t < \infty.
\]
The relation between $\phi_I$ and $\phi_H$ at any time $t$ is given by

$$\phi_I(x, t) = U(t) \phi_H(x, t) U^{-1}(t).$$  \hfill (176)

As $t \to \infty$ the identifications of eqs. (174) and (175) yield

$$\phi_{\text{in}} = U(\infty) \phi_{\text{out}} U^\dagger(\infty).$$  \hfill (177)

From the definition of the $S$-matrix, Eq. (164) we then read off that

$$\lim_{t \to \infty} U(t) = S.$$  \hfill (178)

We have thus derived a formal expression for the $S$-matrix in terms of the operator $U(t)$, which tells us how operators and state vectors deviate from the free theory at time $t$, measured relative to $t_0 = -\infty$, i.e. long before the interaction process.

An important boundary condition for $U(t)$ is

$$\lim_{t \to -\infty} U(t) = 1.$$  \hfill (179)

What we mean here is the following: the operator $U$ actually describes the evolution relative to some initial time $t_0$, which we will normally suppress, i.e. we write $U(t)$ instead of $U(t, t_0)$. We regard $t_0$ merely as a time label and fix it at $-\infty$, where the interaction vanishes. Equation (179) then simply states that $U$ becomes unity as $t \to t_0$, which means that in this limit there is no distinction between Heisenberg and Dirac fields.

Using the definition of $U(t)$, Eq. (172), it is an easy exercise to derive the equation of motion for $U(t)$:

$$i \frac{d}{dt} U(t) = H_{\text{int}}(t) U(t), \quad H_{\text{int}}(t) = e^{iH_0 t} H_{\text{int}} e^{-iH_0 t}.$$  \hfill (180)

The time-dependent operator $H_{\text{int}}(t)$ is defined in the interaction picture, and depends on the fields $\phi_{\text{in}}, \pi_{\text{in}}$ in the “in” basis. Let us now solve the equation of motion for $U(t)$ with the boundary condition $\lim_{t \to -\infty} U(t) = 1$. Integrating Eq. (180) gives

$$\int_{-\infty}^{t} \frac{d}{dt_1} U(t_1) dt_1 = -i \int_{-\infty}^{t} H_{\text{int}}(t_1) U(t_1) dt_1$$

$$U(t) - U(-\infty) = -i \int_{-\infty}^{t} H_{\text{int}}(t_1) U(t_1) dt_1$$

$$\Rightarrow U(t) = 1 - i \int_{-\infty}^{t} H_{\text{int}}(t_1) U(t_1) dt_1.$$  \hfill (181)

The right-hand side still depends on $U$, but we can substitute our new expression for $U(t)$ into the integrand, which gives

$$U(t) = 1 - i \int_{-\infty}^{t} H_{\text{int}}(t_1) \left\{ 1 - i \int_{-\infty}^{t_1} H_{\text{int}}(t_2) U(t_2) dt_2 \right\} dt_1$$

$$= 1 - i \int_{-\infty}^{t} H_{\text{int}}(t_1) dt_1 - \int_{-\infty}^{t} dt_1 H_{\text{int}}(t_1) \int_{-\infty}^{t_1} dt_2 H_{\text{int}}(t_2) U(t_2),$$  \hfill (182)
where \( t_2 < t_1 < t \). This procedure can be iterated further, so that the \( n \)th term in the sum is
\[
(-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n H_{\text{int}}(t_1) H_{\text{int}}(t_2) \cdots H_{\text{int}}(t_n).
\]
(183)

This iterative solution could be written in much more compact form, were it not for the fact that the upper integration bounds were all different, and that the ordering \( t_n < t_{n-1} < \cdots < t_1 < t \) had to be obeyed. Time ordering is an important issue, since one has to ensure that the interaction Hamiltonians act at the proper time, thereby ensuring the causality of the theory. By introducing the time-ordered product of operators, one can use a compact notation, such that the resulting expressions still obey causality. The time-ordered product of two fields \( \phi(t_1) \) and \( \phi(t_2) \) is defined as
\[
T\{ \phi(t_1) \phi(t_2) \} = \begin{cases} 
\phi(t_1)\phi(t_2) & t_1 > t_2 \\
\phi(t_2)\phi(t_1) & t_1 < t_2 
\end{cases}
\equiv \theta(t_1 - t_2) \phi(t_1)\phi(t_2) + \theta(t_2 - t_1) \phi(t_2)\phi(t_1),
\]
(184)

where \( \theta \) denotes the step function. The generalisation to products of \( n \) operators is obvious. Using time ordering for the \( n \)th term of Eq.(183) we obtain
\[
\frac{(-i)^n}{n!} \prod_{i=1}^{n} \int_{-\infty}^{t} dt_i T\{ H_{\text{int}}(t_1) H_{\text{int}}(t_2) \cdots H_{\text{int}}(t_n) \}.
\]
(185)

Here we have replaced each upper limit of integration with \( t \). Each specific ordering in the time-ordered product gives a term identical to eq. (183), where applying the \( T \) operator corresponds to setting the upper limit of integration to the relevant \( t_i \) in each integral. However, we have overcounted by a factor \( n! \), corresponding to the number of ways of ordering the fields in the time ordered product. Thus one must divide by \( n! \) as shown. We may recognise eq. (185) as the \( n \)th term in the series expansion of an exponential, and thus can finally rewrite the solution for \( U(t) \) in compact form as
\[
U(t) = T \exp \left\{ -i \int_{-\infty}^{t} H_{\text{int}}(t') dt' \right\},
\]
(186)

where the “\( T \)” in front ensures the correct time ordering.

4.3 \( S \)-matrix and Green’s functions

The \( S \)-matrix, which relates the “in” and “out” fields before and after the scattering process, can be written as
\[
S = 1 + iT,
\]
(187)

where \( T \) is commonly called the \( T \)-matrix. The fact that \( S \) contains the unit operator means that also the case where none of the particles scatter is encoded in \( S \). On the other hand, the non-trivial case is described by the \( T \)-matrix, and this is what we are interested in. So far we have derived an expression for the \( S \)-matrix in terms of the interaction Hamiltonian, and we could use this in principle to calculate scattering processes. However, there is a slight complication owing to the fact that the vacuum of the free theory is not the same as the true vacuum of the full, interacting
theory. Instead, we will follow the approach of Lehmann, Symanzik and Zimmerman, which relates
the S-matrix to n-point Green’s functions

\[ G_n(x_1, \ldots, x_n) = \langle 0 | T(\phi(x_1) \ldots \phi(x_n)) | 0 \rangle \]  

(188)
i.e. vacuum expectation values of Heisenberg fields. We will see later how to calculate these in
terms of vacuum expectation values of “in” fields (i.e. in the free theory).
In order to relate S-matrix elements to Green’s functions, we have to express the “in/out”-states
in terms of creation operators \( a_{\text{in/out}}^\dagger \) and the vacuum, then express the creation operators by the
fields \( \phi_{\text{in/out}} \), and finally use the time evolution to connect those with the fields \( \phi \) in our Lagrangian.

Let us consider again the scattering process depicted in Fig. 4. The S-matrix element in this case is

\[ S_{\text{fi}} = \left\langle k_1, k_2, \ldots, k_n; \text{out} \left| p_1, p_2; \text{in} \right. \right\rangle \]
\[ = \left\langle k_1, k_2, \ldots, k_n; \text{out} \left| a_{\text{in}}^\dagger(p_1) p_2; \text{in} \right. \right\rangle , \]  

(189)

where \( a_{\text{in}}^\dagger \) is the creation operator pertaining to the “in” field \( \phi_{\text{in}} \). Our task is now to express \( a_{\text{in}}^\dagger \)
in terms of \( \phi_{\text{in}} \), and repeat this procedure for all other momenta labelling our Fock states.

The following identities will prove useful

\[ a^\dagger(p) = i \int d^3 x \left\{ (\partial_0 e^{-ip \cdot x}) \phi(x) - e^{-ip \cdot x} (\partial_0 \phi(x)) \right\} \]
\[ = -i \int d^3 x e^{-ip \cdot x} \overleftarrow{\partial_0} \phi(x), \]
\[ \hat{a}(p) = -i \int d^3 x \left\{ (\partial_0 e^{ip \cdot x}) \phi(x) - e^{ip \cdot x} (\partial_0 \phi(x)) \right\} \]
\[ = i \int d^3 x e^{ip \cdot x} \overrightarrow{\partial_0} \phi(x). \]  

(190)

The S-matrix element can then be rewritten as

\[ S_{\text{fi}} = -i \int d^3 x_1 e^{-ip_1 \cdot x_1} \overleftarrow{\partial_0} \left\{ k_1, \ldots, k_n; \text{out} \left| \phi_{\text{in}}(x_1) \right| p_2; \text{in} \right\} \]
\[ = -i \lim_{t_1 \to -\infty} \int d^3 x_1 e^{-ip_1 \cdot x_1} \overleftarrow{\partial_0} \left\{ k_1, \ldots, k_n; \text{out} \left| \phi(x_1) \right| p_2; \text{in} \right\} , \]  

(192)

where in the last line we have used Eq. (156) to replace \( \phi_{\text{in}} \) by \( \phi \). We can now rewrite \( \lim_{t_1 \to -\infty} \) using the following identity, which holds for an arbitrary, differentiable function \( f(t) \), whose limit \( t \to \pm \infty \) exists:

\[ \lim_{t \to -\infty} f(t) = \lim_{t \to +\infty} f(t) - \int_{-\infty}^{+\infty} \frac{df}{dt} dt. \]  

(193)

The S-matrix element then reads

\[ S_{\text{fi}} = -i \lim_{t_1 \to +\infty} \int d^3 x_1 e^{-ip_1 \cdot x_1} \overleftarrow{\partial_0} \left\{ k_1, \ldots, k_n; \text{out} \left| \phi(x_1) \right| p_2; \text{in} \right\} \]
\[ + i \int_{-\infty}^{+\infty} dt_1 \frac{\partial}{\partial t_1} \left\{ \int d^3 x_1 e^{-ip_1 \cdot x_1} \overleftarrow{\partial_0} \left\{ k_1, \ldots, k_n; \text{out} \left| \phi(x_1) \right| p_2; \text{in} \right\} \right\} . \]  

(194)
The first term in this expression involves \( \lim_{t_1 \to +\infty} \phi = \phi_{\text{out}} \), which gives rise to a contribution

\[
\propto \left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \right| a^+_{\text{out}}(\mathbf{p}_1) \left| \mathbf{p}_2; \text{in} \right\rangle. \tag{195}
\]

This is non-zero only if \( \mathbf{p}_1 \) is equal to one of \( \mathbf{k}_1, \ldots, \mathbf{k}_n \). This, however, means that the particle with momentum \( \mathbf{p}_1 \) does not scatter, and hence the first term does not contribute to the \( T \)-matrix of Eq. (187). We are then left with the following expression for \( S_{\text{fi}} \):

\[
S_{\text{fi}} = -i \int d^4x_1 \left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \right| \partial_0 \left\{ (\partial_0 e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1}) \phi(x_1) - e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} (\partial_0 \phi(x_1)) \right\} \left| \mathbf{p}_2; \text{in} \right\rangle. \tag{196}
\]

The time derivatives in the integrand can be worked out:

\[
\partial_0 \left\{ (\partial_0 e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1}) \phi(x_1) - e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} (\partial_0 \phi(x_1)) \right\}
= - [E(\mathbf{p}_1)]^2 e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} \phi(x_1) - e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} \partial_0^2 \phi(x_1)
= - \left\{ (-\nabla^2 + m^2) e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} \right\} \phi(x_1) + e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} \partial_0^2 \phi(x_1), \tag{197}
\]

where we have used that \( -\nabla^2 e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} = \mathbf{p}_1^2 e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} \). For the \( S \)-matrix element one obtains

\[
S_{\text{fi}} = i \int d^4x_1 e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} \left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \right| \partial_0^2 - \nabla^2 + m^2 \phi(x_1) \left| \mathbf{p}_2; \text{in} \right\rangle
= i \int d^4x_1 e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} (\Box + m^2) \left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \right| \phi(x_1) \left| \mathbf{p}_2; \text{in} \right\rangle, \tag{198}
\]

where we have used integration by parts twice so that \( \nabla^2 \) acts on \( \phi(x_1) \) rather than on \( e^{-i\mathbf{p}_1 \cdot \mathbf{x}_1} \). What we have obtained after this rather lengthy step of algebra is an expression in which the (Heisenberg) field operator is sandwiched between Fock states, one of which has been reduced to a one-particle state. We can now successively eliminate all momentum variables from the Fock states, by repeating the procedure for the momentum \( \mathbf{p}_2 \), as well as the \( n \) momenta of the “out” state. The final expression for \( S_{\text{fi}} \) is

\[
S_{\text{fi}} = (i)^{n+2} \int d^4x_1 \int d^4x_2 \int d^4y_1 \cdots \int d^4y_n e^{(-i\mathbf{p}_1 \cdot \mathbf{x}_1 - i\mathbf{p}_2 \cdot \mathbf{x}_2 + ik_1 \cdot y_1 + \cdots + ik_n \cdot y_n)}
\times (\Box + m^2) (\Box + m^2) (\Box + m^2) \cdots (\Box + m^2)
\times \left\langle 0; \text{out} \right| T\{\phi(y_1) \cdots \phi(y_n) \phi(x_1) \phi(x_2)\} \left| 0; \text{in} \right\rangle, \tag{199}
\]

where the time-ordering inside the vacuum expectation value (VEV) ensures that causality is obeyed. The above expression is known as the \textit{Lehmann-Symanzik-Zimmermann (LSZ) reduction formula}. It relates the formal definition of the scattering amplitude to a vacuum expectation value of time-ordered fields. Since the vacuum is uniquely the same for “in/out”, the VEV in the LSZ formula for the scattering of two initial particles into \( n \) particles in the final state is recognised as the \( (n + 2) \)-point Green’s function:

\[
G_{n+2}(y_1, y_2, \ldots, y_n, x_1, x_2) = \left\langle 0 \left| T\{\phi(y_1) \cdots \phi(y_n) \phi(x_1) \phi(x_2)\} \right| 0 \right\rangle. \tag{200}
\]

You will note that we still have not calculated or evaluated anything, but merely rewritten the expression for the scattering matrix elements. Nevertheless, the LSZ formula is of tremendous
importance and a central piece of QFT. It provides the link between fields in the Lagrangian and the scattering amplitude $S_{2 \rightarrow 2}$, which yields the cross section, measurable in an experiment. Up to here no assumptions or approximations have been made, so this connection between physics and formalism is rather tight. It also illustrates a profound phenomenon of QFT and particle physics: the scattering properties of particles, in other words their interactions, are encoded in the vacuum structure, i.e. the vacuum is non-trivial!

4.4 How to compute Green’s functions

Of course, in order to calculate cross sections, we need to compute the Green’s functions. Alas, for any physically interesting and interacting theory this cannot be done exactly, contrary to the free theory discussed earlier. Instead, approximation methods have to be used in order to simplify the calculation, while hopefully still giving reliable results. Or one reformulates the entire QFT as a lattice field theory, which in principle allows to compute Green’s functions without any approximations (in practice this still turns out to be a difficult task for physically relevant systems). This is what many theorists do for a living. But the formalism stands, and if there are discrepancies between theory and experiments, one “only” needs to check the accuracy with which the Green’s functions have been calculated or measured, before approving or discarding a particular Lagrangian.

In the next section we shall discuss how to compute the Green’s function of scalar field theory in perturbation theory. Before we can tackle the actual computation, we must take a further step. Let us consider the $n$-point Green’s function

$$G_n(x_1, \ldots, x_n) = \langle 0| T\{\phi(x_1) \cdots \phi(x_n)\}|0\rangle.$$  \hfill (201)

The fields $\phi$ which appear in this expression are Heisenberg fields, whose time evolution is governed by the full Hamiltonian $H_0 + H_{\text{int}}$. In particular, the $\phi$’s are not the $\phi_{\text{in}}$’s. We know how to handle the latter, because they correspond to a free field theory, but not the former, whose time evolution is governed by the interacting theory, whose solutions we do not know. Let us thus start to isolate the dependence of the fields on the interaction Hamiltonian. Recall the relation between the Heisenberg fields $\phi(t)$ and the “in”-fields$^4$

$$\phi(t) = U^{-1}(t) \phi_{\text{in}}(t) U(t).$$  \hfill (202)

We now assume that the fields are properly time-ordered, i.e. $t_1 > t_2 > \ldots > t_n$, so that we can forget about writing $T(\cdots)$ everywhere. After inserting Eq. (202) into the definition of $G_n$ one obtains

$$G_n = \langle 0| U^{-1}(t_1) \phi_{\text{in}}(t_1) U(t_1) U^{-1}(t_2) \phi_{\text{in}}(t_2) U(t_2) \cdots$$
$$\times U^{-1}(t_n) \phi_{\text{in}}(t_n) U(t_n) |0\rangle.$$  \hfill (203)

Now we introduce another time label $t$ such that $t \gg t_1$ and $-t \ll t_1$. For the $n$-point function we now obtain

$$G_n = \langle 0| U^{-1}(t) \left\{ U(t) U^{-1}(t_1) \phi_{\text{in}}(t_1) U(t_1) U^{-1}(t_2) \phi_{\text{in}}(t_2) U(t_2) \cdotsight.$$
$$\times U^{-1}(t_n) \phi_{\text{in}}(t_n) U(t_n) U^{-1}(-t) \right\} U(-t) |0\rangle.$$  \hfill (204)

$^4$Here and in the following we suppress the spatial argument of the fields for the sake of brevity.
The expression in curly braces is now time-ordered by construction. An important observation at this point is that it involves pairs of \( U \) and its inverse, for instance
\[
U(t)U^{-1}(t_1) \equiv U(t, t_1).
\]

One can easily convince oneself that \( U(t, t_1) \) provides the net time evolution from \( t_1 \) to \( t \). We can now write \( G_n \) as
\[
G_n = \langle 0 | U^{-1}(t) T \{ \phi_{\text{in}}(t_1) \cdots \phi_{\text{in}}(t_n) U(t, t_1) U(t_1, t_2) \cdots U(t_{n-1}, t) \} U(-t) | 0 \rangle,
\]
where we have used the fact that we may commute the \( U \) operators within the time-ordered product.

Let us now take \( t \to \infty \). The relation between \( U(t) \) and the \( S \)-matrix Eq. (178), as well as the boundary condition Eq. (179) tell us that
\[
\lim_{t \to \infty} U(-t) = 1, \quad \lim_{t \to \infty} U(t, -t) = S,
\]
which can be inserted into the above expression. We still have to work out the meaning of \( \langle 0 | U^{-1}(\infty) \rangle \) in the expression for \( G_n \). In a paper by Gell-Mann and Low it was argued that the time evolution operator must leave the vacuum invariant (up to a phase), which justifies the ansatz
\[
\langle 0 | U^{-1}(\infty) \rangle = K \langle 0 | 0 \rangle,
\]
with \( K \) being the phase\(^5\). Multiplying this relation with \( | 0 \rangle \) from the right gives
\[
\langle 0 | U^{-1}(\infty) | 0 \rangle = K \langle 0 | 0 \rangle = K.
\]

Furthermore, Gell-Mann and Low showed that
\[
\langle 0 | U^{-1}(\infty) | 0 \rangle = \frac{1}{\langle 0 | U(\infty) | 0 \rangle},
\]
which implies
\[
K = \frac{1}{\langle 0 | S | 0 \rangle}.
\]

After inserting all these relations into the expression for \( G_n \) we obtain
\[
G_n(x_1, \ldots, x_n) = \frac{\langle 0 | T \{ \phi_{\text{in}}(x_1) \cdots \phi_{\text{in}}(x_n) S \} | 0 \rangle}{\langle 0 | S | 0 \rangle}.
\]

The \( S \)-matrix is given by
\[
S = T \exp \left\{ -i \int_{-\infty}^{+\infty} H_{\text{int}}(t) \, dt \right\}, \quad H_{\text{int}} = H_{\text{int}}(\phi_{\text{in}}, \pi_{\text{in}}),
\]
and thus we have finally succeeded in expressing the \( n \)-point Green’s function exclusively in terms of the “in”-fields. This completes the derivation of a relation between the general definition of the scattering amplitude \( S_{\text{fi}} \) and the VEV of time-ordered “in”-fields. This has been a long and

\(^5\)As hinted at earlier, \( K \) relates the vacuum of the free theory to the true vacuum of the interacting theory.
technical discussion, but the main points are the following:

Scattering probabilities are related to $S$-matrix elements. To calculate $S$-matrix elements for an $n$ particle scattering process, one must first calculate the $n$ particle Green’s function (eq. (212)). Then one plugs this into the LSZ formula (eq. (199)).

In fact, the Green’s functions cannot be calculated exactly using eq. (212). Instead, one can only obtain answers in the limit in which the interaction strength $\lambda$ is small. This is the subject of the following sections.

## 5 Perturbation Theory

In this section we are going to calculate the Green’s functions of scalar quantum field theory explicitly. We will specify the interaction Lagrangian in detail and use an approximation known as perturbation theory. At the end we will derive a set of rules, which represent a systematic prescription for the calculation of Green’s functions, and can be easily generalised to apply to other, more complicated field theories. These are the famous Feynman rules.

We start by making a definite choice for the interaction Lagrangian $L_{\text{int}}$. Although one may think of many different expressions for $L_{\text{int}}$, one has to obey some basic principles: firstly, $L_{\text{int}}$ must be chosen such that the potential it generates is bounded from below – otherwise the system has no ground state. Secondly, our interacting theory should be renormalisable. Despite being of great importance, the second issue will not be addressed in these lectures. The requirement of renormalisability arises because the non-trivial vacuum, much like a medium, interacts with particles to modify their properties. Moreover, if one computes quantities like the energy or charge of a particle, one typically obtains a divergent result\(^6\). There are classes of quantum field theories, called renormalisable, in which these divergences can be removed by suitable redefinitions of the fields and the parameters (masses and coupling constants).

For our theory of a real scalar field in four space-time dimensions, it turns out that the only interaction term which leads to a renormalisable theory must be quartic in the fields. Thus we choose

$$L_{\text{int}} = -\frac{\lambda}{4!} \phi^4(x), \quad (214)$$

where the coupling constant $\lambda$ describes the strength of the interaction between the scalar fields, much like, say, the electric charge describing the strength of the interaction between photons and electrons. The factor $4!$ is for later convenience. The full Lagrangian of the theory then reads

$$L = L_0 + L_{\text{int}} = \frac{1}{2} \left( \partial_\mu \phi \right)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4, \quad (215)$$

and the explicit expressions for the interaction Hamiltonian and the $S$-matrix are

$$H_{\text{int}} = -L_{\text{int}}, \quad H_{\text{int}} = \frac{\lambda}{4!} \int d^4 x \phi^4_{\text{in}}(x, t)$$

$$S = T \exp \left\{ -i \frac{\lambda}{4!} \int d^4 x \phi^4_{\text{in}}(x) \right\}. \quad (216)$$

\(^6\)This is despite the subtraction of the vacuum energy discussed earlier.
The normal-ordered product is Wick's theorem for the case of two fields:

\[ \sum_{r=0}^{\infty} \left( -\frac{i\lambda}{4!} \right)^r \frac{1}{r!} \left\langle 0 \left| T \left\{ \phi_{m_1}(x_1) \cdots \phi_{m_n}(x_n) \left( \int d^4 y \phi_{m_i}^4(y) \right)^r \right\} \right| 0 \right\rangle. \] (217)

This expression cannot be dealt with as it stands. In order to evaluate it we must expand \( G_n \) in powers of the coupling \( \lambda \) and truncate the series after a finite number of terms. This only makes sense if \( \lambda \) is sufficiently small. In other words, the interaction Lagrangian must act as a small perturbation on the system. As a consequence, the procedure of expanding Green’s functions in powers of the coupling is referred to as perturbation theory. We will see that there is a natural diagrammatic representation of this expansion (Feynman diagrams). First, we need to know how to calculate the vacuum expectation values of time ordered products. This is the subject of the next section.

5.1 Wick’s Theorem

The \( n \)-point Green’s function in Eq. (217) involves the time-ordered product over at least \( n \) fields. There is a method to express VEV’s of \( n \) fields, i.e. \( \left. g(x_1) \cdots g(x_n) \right| 0 \rangle \) in terms of VEV’s involving two fields only. This is known as Wick’s theorem.

Let us for the moment ignore the subscript “in” and return to the definition of normal-ordered fields. The normal-ordered product \( \langle \phi(x_1) \phi(x_2) \rangle \) differs from \( \langle \phi(x_1) \phi(x_2) \rangle \) by the vacuum expectation value, i.e.

\[ \phi(x_1) \phi(x_2) = \langle \phi(x_1) \phi(x_2) \rangle + \langle 0 | \phi(x_1) \phi(x_2) | 0 \rangle. \] (218)

We are now going to combine normal-ordered products with time ordering. The time-ordered product \( T \{ \phi(x_1) \phi(x_2) \} \) is given by

\[
T \{ \phi(x_1) \phi(x_2) \} = \phi(x_1) \phi(x_2) \theta(t_1 - t_2) + \phi(x_2) \phi(x_1) \theta(t_2 - t_1) \\
= : \phi(x_1) \phi(x_2) : \left( \theta(t_1 - t_2) + \theta(t_2 - t_1) \right) \\
+ \langle 0 | \phi(x_1) \phi(x_2) \theta(t_1 - t_2) + \phi(x_2) \phi(x_1) \theta(t_2 - t_1) | 0 \rangle. \] (219)

Here we have used the important observation that

\[ : \phi(x_1) \phi(x_2) : = : \phi(x_2) \phi(x_1) :, \] (220)

which means that normal-ordered products of fields are automatically time-ordered.\(^7\) Equation (219) is Wick’s theorem for the case of two fields:

\[ T \{ \phi(x_1) \phi(x_2) \} = : \phi(x_1) \phi(x_2) : + \langle 0 | T \{ \phi(x_1) \phi(x_2) \} | 0 \rangle. \] (221)

For the case of three fields, Wick’s theorem yields

\[
T \{ \phi(x_1) \phi(x_2) \phi(x_3) \} = : \phi(x_1) \phi(x_2) \phi(x_3) : + : \phi(x_1) : \langle 0 | T \{ \phi(x_2) \phi(x_3) \} | 0 \rangle \\
+ : \phi(x_2) \langle 0 | T \{ \phi(x_1) \phi(x_3) \} | 0 \rangle + : \phi(x_3) : \langle 0 | T \{ \phi(x_1) \phi(x_2) \} | 0 \rangle. \] (222)

\(^7\)The reverse is, however, not true!
At this point the general pattern becomes clear: any time-ordered product of fields is equal to its normal-ordered version plus terms in which pairs of fields are removed from the normal-ordered product and sandwiched between the vacuum to form 2-point functions. Then one sums over all permutations. Without proof we give the expression for the general case of \( n \) fields (\( n \) even):

\[
T\{\phi(x_1) \cdots \phi(x_n)\} = \\
: \phi(x_1) \cdots \phi(x_n) : \\
+ : \phi(x_1) \cdots \phi(x_i) \cdots \phi(x_j) \cdots \phi(x_n) : \langle 0 | T\{\phi(x_i)\phi(x_j)\} | 0 \rangle + \text{perms.} \\
+ : \phi(x_1) \cdots \phi(x_i) \cdots \phi(x_j) \cdots \phi(x_k) \cdots \phi(x_i) \cdots \phi(x_n) : \\
\times \langle 0 | T\{\phi(x_i)\phi(x_j)\} | 0 \rangle \langle 0 | T\{\phi(x_k)\phi(x_i)\} | 0 \rangle + \text{perms.} \\
+ \ldots + \\
+ \langle 0 | T\{\phi(x_1)\phi(x_2)\} | 0 \rangle \langle 0 | T\{\phi(x_3)\phi(x_4)\} | 0 \rangle \cdots \langle 0 | T\{\phi(x_{n-1})\phi(x_n)\} | 0 \rangle \\
+ \text{perms.} \tag{223}
\]

The symbol \( \hat{\phi}(x_i) \) indicates that \( \phi(x_i) \) has been removed from the normal-ordered product. Let us now go back to \( \langle 0 | T\{\phi(x_1) \cdots \phi(x_n)\} | 0 \rangle \). If we insert Wick’s theorem, then we find that only the contribution in the last line of Eq. (223) survives: by definition the VEV of a normal-ordered product of fields vanishes, and it is precisely the last line of Wick’s theorem in which no normal-ordered products are left. The only surviving contribution is that in which all fields have been paired or “contracted”. Sometimes a contraction is represented by the notation:

\[
\phi(x_i)\phi(x_j) \equiv \langle 0 | T\{\phi(x_i)\phi(x_j)\} | 0 \rangle, \tag{224}
\]

i.e. the pair of fields which is contracted is joined by the braces. Wick’s theorem can now be rephrased as

\[
\langle 0 | T\{\phi(x_1) \cdots \phi(x_n)\} | 0 \rangle = \text{sum of all possible contractions of } n \text{ fields.} \tag{225}
\]

An example of this result is the 4-point function

\[
\langle 0 | T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} | 0 \rangle = \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\
+ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) + \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \tag{226}
\]

5.2 The Feynman propagator

Using Wick’s Theorem one can relate any \( n \)-point Green’s functions to an expression involving only 2-point functions. Let us have a closer look at

\[
G_2(x, y) = \langle 0 | T\{\phi_{\text{in}}(x)\phi_{\text{in}}(y)\} | 0 \rangle. \tag{227}
\]
We can now insert the solution for $\phi$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. If we assume $t_x > t_y$ then $G_2(x, y)$ can be written as

$$G_2(x, y) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{4E(p)E(q)} e^{-ip \cdot x + iq \cdot y} \left\{ \hat{a}^\dagger(\mathbf{p}) e^{ip \cdot y} + \hat{a}(\mathbf{q}) e^{-iq \cdot y} \right\} 0.$$  

This shows that $G_2$ can be interpreted as the amplitude for a meson which is created at $y$ and destroyed again at point $x$. We can now replace $\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q})$ by its commutator:

$$G_2(x, y) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{2E(\mathbf{p})} e^{-ip \cdot (x-y)} \left\{ \hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{q}) \right\} 0,$$

and the general result, after restoring time-ordering, reads

$$G_2(x, y) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{2E(\mathbf{p})} \left( e^{-ip \cdot (x-y)} \theta(t_x - t_y) + e^{ip \cdot (x-y)} \theta(t_y - t_x) \right).$$

Furthermore, using contour integration one can show that this expression can be rewritten as a 4-dimensional integral

$$G_2(x, y) = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}.$$  

where $\epsilon$ is a small parameter which ensures that $G_2$ does not develop a pole. This calculation has established that $G_2(x, y)$ actually depends only on the difference $(x - y)$. Equation (231) is called the Feynman propagator $G_F(x - y)$:

$$G_F(x - y) \equiv \langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}.$$  

The Feynman propagator is a Green’s function of the Klein-Gordon equation, i.e. it satisfies

$$(\Box_x + m^2) G_F(x - y) = -i\delta^4(x - y),$$

and describes the propagation of a meson between the space-time points $x$ and $y$.

### 5.3 Two-particle scattering to $O(\lambda)$

Let us now consider a scattering process in which two incoming particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ scatter into two outgoing ones with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, as shown in Fig. 5. The $S$-matrix element in this case is

$$S_{\text{in}} = \langle \mathbf{k}_1, \mathbf{k}_2; \text{out} | \mathbf{p}_1, \mathbf{p}_2; \text{in} \rangle = \langle \mathbf{k}_1, \mathbf{k}_2; \text{in} | S | \mathbf{p}_1, \mathbf{p}_2; \text{in} \rangle,$$
Figure 5: Scattering of two initial particles with momenta $p_1$ and $p_2$ into 2 particles with momenta $k_1$ and $k_2$.

and $S = 1 + iT$. The LSZ formula Eq. (199) tells us that we must compute $G_4$ in order to obtain $S_{fi}$. Let us work out $G_4$ in powers of $\lambda$ using Wick’s theorem.

Suppressing the subscripts “in” from now on, the expression we have to evaluate order by order in $\lambda$ is

$$G_4(x_1, \ldots, x_4) = \sum_{r=0}^{\infty} \left( -\frac{i\lambda}{4!} \right)^r \frac{1}{r!} \left\langle 0 \left| T\left\{ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \left( \int d^4y \phi^4(y) \right)^r \right\} \right| 0 \right\rangle,$$

At $O(\lambda^0)$, the denominator is 1, and the numerator gives

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle = G_F(x_1 - x_2) G_F(x_3 - x_4) + G_F(x_1 - x_3) G_F(x_2 - x_4) + G_F(x_1 - x_4) G_F(x_2 - x_3),$$

where we have used Wick’s theorem. We may represent this graphically as follows:

But this is the same answer as if we had set $\lambda = 0$, so $O(\lambda^0)$ does not describe scattering and hence is not a contribution to the $T$-matrix.

The first non-trivial scattering happens at $O(\lambda)$. For example, the expansion of the above formula includes the contribution (from the numerator)

$$-\frac{i\lambda}{4!}\langle 0|T[\phi(x_1)\ldots\phi(x_4)] \int d^4y \phi^4(y)|0\rangle = -\frac{i\lambda}{4!} \int d^4y \int d^4y G_F(x_1 - y) G_F(x_2 - y) G_F(x_3 - y) G_F(x_4 - y).$$
where the $4!$ inside the integral arises from all possible contractions in Wick’s theorem. This has the graphical representation

$$-i\lambda \int d^4 y$$

where each line corresponds to a propagator, and we have assigned a vertex to each space-time point. Also at this order, we have the graphs

$$\ldots$$

We will see later on that neither of these graphs contributes to the $S$-matrix element (after substituting the Green’s function into the LSZ formula of eq. (199)), as they are not fully connected. By this we mean that not all external particle vertices are connected to the same graph. At yet higher orders, we may have graphs which involve fully connected pieces, dressed by additional “vacuum bubbles” (such as that which is sitting in the middle of the right-most figure above). These vacuum bubbles are cancelled by the denominator in eq. (212) which, given that it contains no external fields, generates all possible vacuum graphs. The presence of these vacuum graphs explains why the vacuum of the interacting theory is different to that of the free theory, as mentioned earlier.

To summarise, the final answer for the scattering amplitude to $O(\lambda)$ is given by Eq. (237).

### 5.4 Graphical representation of the Wick expansion: Feynman rules

We have already encountered the graphical representation of the expansion of Green’s functions in perturbation theory after applying Wick’s theorem. It is possible to formulate a simple set of rules which allow us to draw the graphs directly without using Wick’s theorem and to write down the corresponding algebraic expressions.

We again consider a neutral scalar field whose Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4.$$  \hspace{1cm} (238)$$

Suppose now that we want to compute the $O(\lambda^m)$ contribution to the $n$-point Green’s function $G_n(x_1, \ldots, x_n)$. This is achieved by going through the following steps:

1. Draw all distinct diagrams with $n$ external lines and $m$ 4-fold vertices:
   - Draw $n$ dots and label them $x_1, \ldots, x_n$ (external points)
   - Draw $m$ dots and label them $y_1, \ldots, y_m$ (vertices)
• Join the dots according to the following rules:
  – only one line emanates from each \( x_i \)
  – exactly four lines run into each \( y_j \)
  – the resulting diagram must be connected, i.e. there must be a continuous path between any two points.

(2) Assign a factor \(-\frac{i\lambda}{4!} \int d^4y_i\) to the vertex at \( y_i \)

(3) Assign a factor \( G_F(x_i - y_j)\) to the line joining \( x_i \) and \( y_j \)

(4) Multiply by the number of contractions \( C \) from the Wick expansion which lead to the same diagram.

These are the Feynman rules for scalar field theory in position space.
Let us look at an example, namely the 2-point function. According to the Feynman rules the contributions up to order \( \lambda^2 \) are as follows:

\[
O(1): \quad x_1 \quad \cdots \quad x_2 = G_F(x_1 - x_2)
\]

\[
O(\lambda): \quad x_1 \quad \bigcirc \quad y \quad x_2 = \frac{i\lambda}{2!} \int d^4y G_F(x_1 - y) G_F(x_2 - y) G_F(0)
\]

\[
O(\lambda^2): \quad x_1 \quad \bigcirc \quad y_2 \quad \bigcirc \quad y_1 \quad x_2 = -\frac{\lambda^2}{2} \int d^4y \int d^4z G_F(x_1 - y) G_F(x_2 - y) G_F(y_1 - y) G_F(0)
\]

\[
O(\lambda^2): \quad x_1 \quad \bigcirc \quad y_1 \quad \bigcirc \quad y_2 \quad x_2 = C \left( -\frac{i\lambda}{4!} \right)^2 \int d^4y_1 d^4y_2 G_F(x_1 - y_1) [G_F(y_1 - y_2)]^3 G_F(y_2 - x_2)
\]

The combinatorial factor for this contribution is worked out as \( C = 4 \times 4! \). Note that the same graph, but with the positions of \( y_1 \) and \( y_2 \) interchanged is topologically distinct. Numerically it has the same value as the above graph, and so the corresponding expression has to be multiplied by a factor 2.

Another contribution at order \( \lambda^2 \) is
This contribution must be discarded, since not all of the points are connected via a continuous line.

### 5.5 Feynman rules in momentum space

It is often simpler to work in momentum space, and hence we will discuss the derivation of Feynman rules in this case. This also reflects what is typically done in scattering experiments (i.e. incoming and outgoing particles have definite momentum). If one works in momentum space, the Green’s functions are related to those in position space by a Fourier transform

\[
\tilde{G}_n(p_1, \ldots, p_n) = \int d^4x_1 \cdots \int d^4x_n e^{ip_1 \cdot x_1 + \cdots + ip_n \cdot x_n} G_n(x_1, \ldots, x_n).
\]  

(239)

The Feynman rules then serve to compute the Green’s function \(\tilde{G}_n(p_1, \ldots, p_n)\) order by order in the coupling.

Let us see how this works for the \(2 \rightarrow 2\) scattering example we considered above. At \(O(\lambda)\) this was given in eq. (237), which we may simplify slightly to

\[
-i\lambda \int d^4y G_F(x_1 - y)G_F(x_2 - y)G_F(x_3 - y)G_F(x_4 - y).
\]  

(240)

We may now substitute in the momentum space form of each propagator (eq. (232)) to give

\[
-i\lambda \int d^4y \left( \prod_{i=1}^4 \int \frac{d^4p_i}{(2\pi)^4} \frac{i}{p_i^2 - m^2 + i\epsilon} \right) e^{-i\sum_i p_i \cdot (x_i - y)}
\]

\[
= -i\lambda(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \left( \prod_{i=1}^4 \int \frac{d^4p_i}{(2\pi)^4} \frac{i}{p_i^2 - m^2 + i\epsilon} \right) e^{-i\sum_i p_i \cdot x_i},
\]

where we have carried out the \(y\) integration in the second line. Substituting this into eq. (239) and carrying out the integrals over each \(x_i\), one finds

\[
\tilde{G}_4(p_1, \ldots, p_n) = -i\lambda(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \left( \prod_{i=1}^4 \int \frac{d^4p_i}{(2\pi)^4} \frac{i}{p_i^2 - m^2 + i\epsilon} \right) (2\pi)^4 \delta(p_1)
\]

\[
= -i\lambda(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \prod_{i} \frac{i}{p_i^2 - m^2 + i\epsilon}
\]

We will not repeat the above derivation for a general Green’s function. Rather, we now state the Feynman rules in momentum space, and the reader may easily verify that the above example is a special case.

**Feynman rules (momentum space)**
(1) Draw all distinct diagrams with $n$ external lines and $m$ 4-fold vertices:

- Assign momenta $p_1, \ldots, p_n$ to the external lines
- Assign momenta $k_j$ to the internal lines

(2) Assign to each external line a factor

$$\frac{i}{p_k^2 - m^2 + i\epsilon}$$

(3) Assign to each internal line a factor

$$\int \frac{d^4k_j}{(2\pi)^4} \frac{i}{k_j^2 - m^2 + i\epsilon}$$

(4) Each vertex contributes a factor

$$-\frac{i\lambda}{4!}(2\pi)^4\delta^4\left(\sum \text{momenta}\right),$$

(the delta function ensures that momentum is conserved at each vertex).

(5) Multiply by the combinatorial factor $C$, which is the number of contractions leading to the same momentum space diagram (note that $C$ may be different from the combinatorial factor for the same diagram considered in position space!)

Alternatively, one may rephrase (4) and (5) as follows:

(4\*) Each vertex carries a factor

$$-i\lambda(2\pi)^4\delta^4\left(\sum \text{momenta}\right),$$

(5\*) Divide by the symmetry factor i.e. the dimension of the group of symmetry transformations that leaves the diagram invariant.

5.6 $S$-matrix and truncated Green’s functions

The final topic in these lectures is the derivation of a simple relation between the $S$-matrix element and a particular momentum space Green’s function, which has its external legs amputated: the so-called truncated Green’s function. This further simplifies the calculation of scattering amplitudes using Feynman rules.

Let us return to the LSZ formalism and consider the scattering of $m$ initial particles (momenta $p_1, \ldots, p_m$) into $n$ final particles with momenta $k_1, \ldots, k_n$. The LSZ formula (eq. (199)) tells us that the $S$-matrix element is given by

$$\langle k_1, \ldots, k_n; \text{out} | p_1, \ldots, p_m; \text{in} \rangle = (i)^{n+m} \int \prod_{i=1}^m d^4x_i \int \prod_{j=1}^n d^4y_j \exp \left\{ -i \sum_{i=1}^m p_i \cdot x_i + i \sum_{j=1}^n k_j \cdot y_j \right\} \times \prod_{i=1}^m (\Box x_i + m^2) \prod_{j=1}^n (\Box y_j + m^2) G_{n+m}(x_1, \ldots, x_m, y_1, \ldots, y_n).$$

(241)
Let us have a closer look at $G_{n+m}(x_1, \ldots, x_m, y_1, \ldots, y_n)$. As shown in Fig. 6 it can be split into Feynman propagators, which connect the external points to the vertices at $z_1, \ldots, z_{n+m}$, and a remaining Green’s function $\overline{G}_{n+m}$, according to

$$G_{n+m} = \int d^4 z_1 \cdots d^4 z_{n+m} G_F(x_1 - z_1) \cdots G_F(y_n - z_{n+m}) \overline{G}_{n+m}(z_1, \ldots, z_{n+m}).$$  \hspace{1cm} (242)

where, perhaps for obvious reasons, $\overline{G}_{n+m}$ is called the truncated Green’s function.

Putting Eq.(242) back into the LSZ expression for the $S$-matrix element, and using that

$$(\Box x + m^2) G_F(x_i - z_i) = -i\delta^4(x_i - z_i)$$

one obtains

$$\langle k_1, \ldots, k_n; \text{out} | p_1, \ldots, p_m; \text{in} \rangle = \int \prod_{i=1}^m d^4 x_i \prod_{j=1}^n d^4 y_j \exp \left\{ -i \sum_{i=1}^m p_i \cdot x_i + i \sum_{j=1}^n k_j \cdot y_j \right\}$$

$$\times (-i)^{n+m} \int d^4 z_1 \cdots d^4 z_{n+m} \delta^4(x_1 - z_1) \cdots \delta^4(y_n - z_{n+m}) \overline{G}_{n+m}(z_1, \ldots, z_{n+m}).$$

After performing all the integrations over the $z_k$’s, the final relation becomes

$$\langle k_1, \ldots, k_n; \text{out} | p_1, \ldots, p_m; \text{in} \rangle$$

$$= \int \prod_{i=1}^m d^4 x_i \prod_{j=1}^n d^4 y_j \exp \left\{ -i \sum_{i=1}^m p_i \cdot x_i + i \sum_{j=1}^n k_j \cdot y_j \right\}$$

$$\times \overline{G}_{n+m}(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

$$\equiv \overline{Y}_{n+m}(p_1, \ldots, p_m, k_1, \ldots, k_n),$$

(245)

where $\overline{Y}_{n+m}$ is the truncated $n+m$-point function in momentum space. This result shows that the scattering matrix element is directly given by the truncated Green’s function in momentum space. In other words, calculating the $S$-matrix is much the same as calculating the Green’s function, but without the free propagators associated with the external legs. Note that this renders zero any graph which is not fully connected - any diagram in which not all external points are connected to the same graph vanishes upon multiplication by the $(p_i^2 + m^2)$ factors. This is what allowed us to neglect such graphs in the previous section.
6 Summary

That completes this introductory look at quantum field theory. Although we did not get as far as some of the more relevant physical applications of QFT, we have looked in detail at what a QFT is, and how the description of scattering amplitudes leads to Feynman diagrams. To recap how we did this:

1. We reviewed the Lagrangian formalism for classical field theory, and also the canonical quantisation approach to quantum mechanics.

2. We constructed the Lagrangian for a relativistic field theory (the free Klein-Gordon field), and applied the techniques of canonical quantisation to this field theory.

3. States in this theory were found to represent particle excitations, such that a particle of momentum \( p \) was found to be a quantum of excitation in the relevant Fourier mode of the field.

4. We then studied the interacting theory, arguing that at initial and final times (when the interaction dies away) we can work with free fields. These were related by an operator \( S \), whose matrix elements represented the transition probability to go from a given initial to a given final state.

5. Using the interaction picture for time evolution, we found an expression for the \( S \) matrix in terms of an evolution operator \( U \), describing how the fields at general time \( t \) deviate from the initial free fields.

6. We also found a formula which related \( S \) matrix elements to \( n \)-particle Green’s functions (vacuum expectation values of time-ordered fields). This was the LSZ formula of eq. (199).

7. We related the Green’s functions involving Heisenberg fields to those involving the “in” fields at time \( t \to -\infty \) (eq. (212)).

8. We then found how to compute these Green’s functions in perturbation theory, valid when the strength of the interaction is weak. This involved having to calculate vacuum expectation values of time-ordered products, for which we could use Wick’s theorem.

9. We developed a graphical representation of Wick’s theorem, which led to simple rules (Feynman rules) for the calculation of Green’s functions in position or momentum space.

10. These can easily be converted to \( S \) matrix elements by truncating the free propagators associated with the external lines.

Needless to say, there are many things we did not have time to talk about. Some of these will be explored by the other courses at this school:

- Here we calculated \( S \)-matrix elements without explaining how to turn these into decay rates or cross-sections, which are the measurable quantities. This is dealt with in the QED / QCD course.
• The Klein-Gordon field involves particles of spin zero, which are bosons. One may also construct field theories for fermions of spin \( \frac{1}{2} \), and vector bosons (spin 1). Physical examples include QED and QCD.

• Fields may have internal symmetries (e.g. local gauge invariance). Again, see the QED / QCD and Standard Model courses.

• Diagrams involving loops are divergent, ultimately leading to infinite renormalisation of the couplings and masses. The renormalisation procedure can only be carried out in certain theories. The Standard Model is one example, but other well-known physical theories (e.g. general relativity) fail this criterion.

• There is an alternative formulation of QFT in terms of path integrals (i.e sums over all possible configurations of fields). This alternative formulation involves some extra conceptual overhead, but allows a much more straightforward derivation of the Feynman rules. More than this, the path integral approach makes many aspects of field theory manifest i.e. is central to our understanding of what a quantum field theory is. This will not be covered at all in this school, but the interested student will find many excellent textbooks on the subject.

There are other areas which are not covered at this school, but nonetheless are indicative of the fact that field theory is still very much an active research area, with many exciting new developments:

• Calculating Feynman diagrams at higher orders is itself a highly complicated subject, and there are a variety of interesting mathematical ideas (e.g. from number theory and complex analysis) involved in current research.

• Sometimes perturbation theory is not well-behaved, in that there are large coefficients at each order of the expansion in the coupling constant. Often the physics of these large contributions can be understood, and summed up to all orders in the coupling. This is known as resummation, and is crucial to obtaining sensible results for many cross-sections, especially in QCD.

• Here we have “solved” for scattering probabilities using a perturbation expansion. It is sometimes possible to numerically solve the theory fully non-perturbatively. Such approaches are known as lattice field theory, due to the fact that one discretizes space and time into a lattice of points. It is then possible (with enough supercomputing power!) to calculate things like hadron masses, which are completely incalculable in perturbation theory.

• Here we set up QFT in Minkowski (flat space). If one attempts to do the same thing in curved space (i.e. a strong gravitational field), many weird things happen that give us tantalising hints of what a quantum field of gravity should look like.

• There are some very interesting recent correspondences between certain limits of certain string theories, and a particular quantum field theory in the strong coupling limit. This has allowed us to gain new insights into nonperturbative field theory from an analytic point of view, and there have been applications in heavy ion physics and even condensed matter systems.

I could go on of course, and many of the more formal developments of current QFT research are perhaps not so interesting to a student in experimental particle physics. However, at the present
time some of the more remarkable and novel extensions to the Standard Model (SUSY, extra dimensions) are not only testable, but are actively being looked for. Thus QFT, despite its age, is very much at the forefront of current research efforts and may yet surprise us!

Acknowledgments

I am very grateful to Chris White and Mrinal Dasgupta for providing a previous set of lecture notes, on which these notes are heavily based.

A Books on QFT

There are numerous textbooks already and a surprisingly high number of new books are appearing all the time. As with anything in theoretical physics, exploring a multitude of approaches to a certain field is encouraged.

In the following list, [1] is said to be a good introductory text and a lot of my colleagues use this one for their introduction to QFT classes. Mark has also put a “try-before-buy” version on his webpage, which is an early version of the entire textbook. You can judge yourself if it’s worth the investment.

My first encounter with QFT was [2]. It’s a very good book that heavily makes use of the Path Integral Formalism (not discussed in these lectures), it also includes topics which are normally not featured in general purpose QFT books (e.g. SUSY, topological aspects). A modern classic is [3], which many use as a standard text. It covers a lot of ground and develops an intuitive approach to QFT (but you aren’t spared the hard bits!). It also touches other areas where QFT finds application (e.g. Statistical Physics). In my opinion, it isn’t very good to look things up because Peskin’s pedagogical approach forces logically-connected topics to be scattered across the text. Unless you are very familiar with the book, it can take ages to find certain things again. My personal favorite by far is [4], probably owing to the authors’ focus on particle theory applications of QFT. But you’ll probably need a bit of exposure to one of the introductory texts to fully appreciate the depth and technical details that the authors have put into it. Yes, it’s expensive (like most of the Graduate-level textbooks), but having a advanced QFT book by a bunch of German authors on your shelf will not go unnoticed by your colleagues. Another good text is [5]. Finally, those who are not faint of heart and who like their field theory from the horse’s mouth may like to consult Weinberg’s monumental three volume set [6].

References


### B Notation and conventions

**4-vectors:**

\[ x^\mu = (x^0, \mathbf{x}) = (t, \mathbf{x}) \]

\[ x_\mu = g_{\mu\nu} x^\nu = (x^0, -\mathbf{x}) = (t, -\mathbf{x}) \]

Metric tensor:

\[ g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

**Scalar product:**

\[ x^\mu x_\mu = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3 \]

\[ = t^2 - \mathbf{x}^2 \]

**Gradient operators:**

\[ \partial^\mu \equiv \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, -\nabla \right) \]

\[ \partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, \nabla \right) \]

d’Alembertian:

\[ \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box \]

**Momentum operator:**

\[ \hat{p}^\mu = i\hbar \partial^\mu = \left( i\hbar \frac{\partial}{\partial t}, -i\hbar \nabla \right) = \left( \hat{E}, \hat{\mathbf{p}} \right) \quad \text{(as it should be)} \]

**\( \delta \)-functions:**

\[ \int d^3 p \, f(\mathbf{p}) \delta^3(\mathbf{p} - \mathbf{q}) = f(\mathbf{q}) \]

\[ \int d^3 x \, e^{-i\mathbf{p} \cdot \mathbf{x}} = (2\pi)^3 \delta^3(\mathbf{p}) \]

\[ \int \frac{d^3 p}{(2\pi)^3} \, e^{-i\mathbf{p} \cdot \mathbf{x}} = \delta^3(\mathbf{x}) \]

(similarly in four dimensions)

**Note:**

\[ \delta(x^2 - x_0^2) = \delta\{ (x-x_0)(x+x_0) \} \]

\[ = \frac{1}{2x} \left\{ \delta(x-x_0) + \delta(x+x_0) \right\} \]
AN INTRODUCTION TO QED & QCD

Dr Elena Accomando (University of Southampton)
This course gives an introduction to the ingredients of gauge theories which are necessary to calculate cross sections for particular processes. The section headings are given below:

Outline of Lectures:

1. Relativistic Quantum Mechanics
2. Spin
3. Relativistic Electromagnetism
4. Coulomb Scattering, $e\mu \rightarrow e\mu$
5. Compton Scattering, $e\gamma \rightarrow e\gamma$
6. Colour
7. Renormalisation

This course runs in parallel with the Quantum Field Theory course, from which we will use some results. Some topics mentioned in this course will be covered in more detail in the Standard Model and Phenomenology courses next week.
Textbooks:

These notes are intended to be self-contained, but only provide a short introduction to a complex and fascinating topic. You may find the following textbooks useful:

3. Peskin and Schröder, *An Introduction to Quantum Field Theory*, ABP.
4. Ryder, *Quantum Field Theory*, CUP.
5. Srednicki, *Quantum Field Theory*, CUP.

The first two are more practical and closer to the spirit of this course while the other contain many more mathematical details. The last one is very recent. If you are particularly interested in (or confused by) a particular topic, I encourage you to take a look at it. If there are other textbooks which you find particularly helpful, please tell me and I will update the list.

These notes are based heavily on the content of previous versions of this course, in particular the 2013 version by Jennifer Smillie. Throughout, we will use “natural units” where $\hbar = c = 1$ and the metric signature $(+−−−)$.

Please email any comments, questions or corrections to a.banfi@sussex.ac.uk.

Andrea Banfi
May 25, 2016
1 Relativistic Quantum Mechanics

In order to describe the dynamics of particles involved in high-energy collisions we must be able to combine the theory of phenomena occurring at the smallest scales, i.e. quantum mechanics, with the description of particles moving close to the speed of light, i.e. special relativity. To do this we must develop wave equations which are relativistically invariant (i.e. invariant under Lorentz transformations). In this section we will derive relativistic equations of motion for scalar particles (spin-0) and particles with spin-1/2.

1.1 The Klein-Gordon Equation

We start with the Hamiltonian for a particle in classical mechanics:

\[ E = \frac{p^2}{2m} + V(x). \]  

(1)

To convert this into a wave equation, we make the replacements \( E \to i\partial_t \) and \( p \to -i\nabla \), so that a plane-wave solution

\[ \phi(t, x) \propto e^{-i(Et-p\cdot x)} = e^{-ip\cdot x} \]  

(2)

has the energy-momentum relation given in eq. (1). Applied to a general wavefunction \( \phi \), a linear superposition of plane waves, this gives

\[ i\partial_t \phi(t, x) = \left( -\frac{1}{2m} \nabla^2 + V(x) \right) \phi(t, x) = H \phi(t, x), \]  

(3)

where \( H \) is the so-called Hamiltonian. We recognise this as the Schrödinger Equation, the cornerstone of Quantum Mechanics. From this form, we can deduce that eq. (3) cannot be relativistically invariant because time appears only through a first-order derivative on the left-hand side while space appears as a second-order derivative on the right-hand side. Yet we know that if we make a Lorentz transformation in the \( x \) direction for example, this would mix the \( x \) and \( t \) components and therefore they cannot have different roles.

The problem with the Schrödinger Equation arose because we started from a non-relativistic energy-momentum relation. Let us then start from the relativistic equation for energy. For a particle with 4-momentum \( p^\mu = (E, p) \) and mass \( m \),

\[ E^2 = m^2 + p^2. \]  

(4)

Again we convert this to an operator equation by setting \( p_\mu = i\partial_\mu \) so that the corresponding wave equation for an arbitrary scalar wavefunction \( \phi(x, t) \) gives

\[ (\partial_t^2 - \nabla^2 + m^2) \phi(t, x) = (\partial_\mu \partial^\mu + m^2) \phi(x) = (\Box + m^2)\phi(x) = 0, \]  

(5)
where we have introduced the four-vector $x^\mu = (t, \mathbf{x})$. This is the “Klein-Gordon equation” which is the equation of motion for a free scalar field. We can explicitly check that this is indeed Lorentz invariant. Under a Lorentz transformation
\[ x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \quad \Rightarrow \quad \partial_\mu \rightarrow \partial'_\mu = (\Lambda^{-1})^\rho_\mu \partial_\rho, \]
(6)
The field $\phi$ is a scalar, i.e. it has the transformation property
\[ \phi(x) \rightarrow \phi'(x') = \phi'(\Lambda x) = \phi(x). \]
(7)
Therefore, in the primed system,
\[ \left( \partial'_\mu \partial'^\mu + m^2 \right) \phi'(x') = \left[ (\Lambda^{-1})^\rho_\mu \partial_\rho (\Lambda^{-1})_\sigma^\nu \partial_\sigma g^{\mu\nu} + m^2 \right] \phi'(\Lambda x) = 0, \]
(8)
and the equation still holds.

### 1.2 The Dirac Equation

The Klein-Gordon equation admits negative-energy solutions, because the energy $E$ appearing in the plane-wave in eq. (2) can have the two values $\pm \sqrt{p^2 + m^2}$. Dirac sought to find an alternative relativistic equation which was linear in $\partial_t$ like the Schrödinger equation (this was an attempt to solve the problem of negative-energy solutions to eq. (5) – in fact he didn’t solve this problem, but a different one). If the equation is linear in $\partial_t$, it must also be linear in $\nabla$ if it is to be invariant under Lorentz transformations. We therefore start with the general form
\[ i \partial_t \psi(t, \mathbf{x}) = (-i\alpha \cdot \nabla + \beta m) \psi(t, \mathbf{x}). \]
(9)
Dirac also required that the solutions of his equation would be a solution of the Klein-Gordon equation as well, or equivalently, the energy relation eq. (4) was the correct energy-momentum relation for plane wave solutions $e^{-i p \cdot x}$ of the Dirac equation. To see what constraints this imposes, we must square eq. (9):
\[ -\partial_t^2 \psi(t, \mathbf{x}) = i \partial_t \left( -i\alpha \cdot \nabla + \beta m \right) \psi(t, \mathbf{x}) \]
\[ = (-i\alpha \cdot \nabla + \beta m)^2 \psi(t, \mathbf{x}) \]
\[ = \left[ -\alpha^i \alpha^j \nabla^i \nabla^j - i (\beta \alpha^i + \alpha^i \beta) m \nabla^i + \beta^2 m^2 \right] \psi(t, \mathbf{x}). \]
(10)
However, the Klein-Gordon equation requires that the right-hand side is equal to $[-\nabla^2 + m^2] \psi(t, \mathbf{x})$ and therefore $\alpha$ and $\beta$ must satisfy
\[ \alpha^i \alpha^j + \alpha^j \alpha^i = \{\alpha^i, \alpha^j\} = 2\delta^{ij}, \quad \beta \alpha^i + \alpha^i \beta = \{\alpha^i, \beta\} = 0, \quad \beta^2 = 1. \]
(11)
If $\alpha^i$ and $\beta$ are just numbers, these equations cannot be solved. Dirac solved them by instead taking $\alpha^i$ and $\beta$ to be $n \times n$ matrices, and $\psi(t, \mathbf{x})$ to be a column vector. Even now, the solution is not immediate. One can show that the conditions in eq. (11) require
\[ \text{Tr} \alpha^i = 0 = \text{Tr} \beta, \]
(12)
and further that the eigenvalues of the above matrices are $\pm 1$. This in turn means that $n$ must be even (do you understand why?). In 2-dimensions, there are still not enough linearly independent matrices to satisfy eq. (11). There do exist solutions in four dimensions. One such solution is

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix},$$

(13)

where $\sigma$ are the usual Pauli matrices and $1_2$ represents the $2 \times 2$ identity matrix. Now we have formed an equation which may be thought of as a square-root of the Klein-Gordon equation, but which is not obviously Lorentz invariant. To show that, we first define the new matrices

$$\gamma^0 = \beta, \quad \gamma = \beta \alpha.$$  

(14)

Then we form $\gamma^\mu = (\gamma^0, \gamma)$ where the $\mu$ is a Lorentz index. Each component is a $4 \times 4$ matrix. In terms of the $\gamma$-matrices, one can write the conditions in eq. (11) in a Lorentz covariant form

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}.$$  

(15)

This is an example of a Clifford algebra. Any matrices satisfying this condition in eq. (15) may be used to construct the Dirac equation. The representation in eqs. (13) and (14) is just one example, known as the Dirac representation. Note, for example, that any other matrices satisfying

$$\alpha'_i = U \alpha_i U^{-1}, \quad \text{and} \quad \beta' = U \beta U^{-1},$$

(16)

where $U$ is a unitary matrix, will also be suitable.

Multiplying through by $\gamma^0$, we may rewrite the eq. (9) in a covariant form as

$$(i\gamma^\mu \partial_\mu - m 1_4)\psi(t, x) = (i\partial - m)\psi(x) = 0,$$  

(17)

where $\alpha$, a vector with a slash, is a short-hand notation for $\gamma^\mu a_\mu$. The equation above is known as the Dirac equation. In momentum space, i.e. after a Fourier transformation, $\partial_\mu \to -ip_\mu$, and the Dirac equation becomes

$$(\gamma^\mu p_\mu - m 1_4)\tilde{\psi}(p) = (\not{p} - m)\tilde{\psi}(p) = 0,$$  

(18)

where $\tilde{\psi}(p)$ is the Fourier transform of a solution of the Dirac equation $\psi(x)$.

We mentioned in passing that $\psi(t, x)$ is a column vector rather than a scalar. This means that it contains more than one degree of freedom. Dirac exploited this property to interpret his equation as the wave equation for spin-1/2 particles, fermions, which can be either spin-up or spin-down. The column vector $\psi$ is known as a Dirac spinor.

Comparing eq. (9) to the Schrödinger equation in eq. (3) gives the Hamiltonian for a free spin-1/2 particle:

$$H_{\text{Dirac}} = -i\alpha \cdot \nabla + \beta m.$$  

(19)
The trace of the Hamiltonian gives the sum of the energy eigenvalues. The condition that the matrices $\alpha$ and $\beta$ are traceless therefore means that the eigenvalues of $H_{\text{Dirac}}$ must sum to zero. Therefore, like the Klein-Gordon equation, also the Dirac equation has negative-energy solutions.

Dirac himself proposed a solution for this problem which became known as the “Dirac sea”. He accepted the existence of negative-energy states, but took the vacuum as the state in which all these states are filled, see fig. 1. There is a conceptual problem with this in that the vacuum has infinite negative charge and energy. However, any observation relies only on energy differences, so this picture can give an acceptable theory.

As the negative-energy states are already full, the Pauli exclusion principle forbids any positive-energy electron to fall into one of the negative-energy states. If instead energy is supplied, an electron is excited from a negative-energy state to a positive-energy state and an “electron-hole” pair is created. The absence of the negative-energy electron, the hole, is interpreted as the presence of of state with positive energy and positive charge, i.e. a positron. Dirac predicted the existence of the positron in 1927 and this particle was discovered five years later.

However, Dirac’s argument only holds for spin-1/2 particles which obey the Pauli exclusion principle. A consistent solution for all particles is provided by Quantum Field Theory in a picture developed by Feynman and Stückelberg, in which positive-energy particles travel only forward in time, whereas negative-energy particles travel only backwards in time. In this way, a negative-energy particle with momentum $p^\mu$, travelling backward in time, is re-interpreted as a positive energy anti-particle with momentum $-p^\mu$ travelling forward in time. Let us see how this picture naturally arises by considering two processes, the scattering $e^- \mu^- \rightarrow e^- \mu^-$, and Compton scattering $e^- \gamma \rightarrow e^- \gamma$. In non-relativistic quantum mechanics, the scattering $e^- \mu^- \rightarrow e^- \mu^-$ corresponds to the scattering of an electron from an external Coulomb potential. This is represented on the left-hand side of fig. 2. The horizontal axis represents the time at which a give elementary process occurs.
In non-relativistic quantum mechanics, scattering happens instantaneously, so that the time $t_1$ at which a photon is emitted by the incoming electron coincides with the time $t_2$ in which it is absorbed by a muon, which stays at rest as a source of a static potential. In quantum field theory the scattering cannot occur instantaneously, because we need to take into account the fact that the photon mediating the scattering travels at the speed of light. The corresponding scattering amplitude is given by the sum of the contributions of the two diagrams on the right-hand side of fig. 2. It is clear that, in the limit in which $c$ can be taken to be infinite, the two diagrams coincide and give the non-relativistic contribution. From the point of view of the electron, the first diagram can be interpreted as the emission of a positive-energy photon at $t = t_1$ that travels forward in time, and is later absorbed by a muon at $t = t_2$. The second diagram has an awkward interpretation from the point of view of the electron, because it corresponds to the emission of a negative-energy photon at $t = t_2$ that travels backwards in time. However, the graph makes perfectly sense if one considers that it is the muon that emits a photon a time $t_1$, which is later reabsorbed by the electron at a time $t_2$. A similar interpretation can be applied to the Compton scattering diagrams in Fig. 3, and clarifies the Feynman and Stückelberg interpretation of negative-energy states. In the left diagram, an electron emits a photon at time $t_1$ and later, at time $t_2$ absorbs another one. In the right-hand diagram it appears as if an electron emits a photon and then travels backwards in time to absorb another photon. Feynman and Stückelberg reasoned instead that the incoming photon split into an electron-positron pair and then at a later time, the positron annihilates the other electron, emitting a photon.

2 Spin

In the previous section, we introduced a Dirac spinor as a solution to the Dirac equation in the form of a column vector. In this section, we will discuss the explicit form of the solutions to the Dirac equation, and verify that they indeed correspond to the wave functions for particles with spin-1/2.
2.1 Plane-Wave Solutions of the Dirac Equation

We begin by seeking plane-wave solutions to the Dirac Equation. Given the $2 \times 2$ block nature of the $\gamma$-matrices, we will start with the form

$$\psi(x) = \begin{pmatrix} \chi(p) \\ \phi(p) \end{pmatrix} e^{-ip \cdot x},$$

where $\chi$ and $\phi$ are two-component spinors. Substituting this into eq. (18) and using eqs. (13) and (14), we find

$$p^0 \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \sigma \cdot p \\ \sigma \cdot p & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix},$$

or equivalently

$$\begin{pmatrix} \sigma \cdot p \end{pmatrix} \phi = (p^0 - m)\chi$$

$$\begin{pmatrix} \sigma \cdot p \end{pmatrix} \chi = (p^0 + m)\phi.$$  \hspace{1cm} (21)

From the identity $(\sigma \cdot p)^2 = p^2$, these equations are only consistent for particles with $p^0 = \pm \sqrt{p^2 + m^2}$ (consistent with having solutions of the Klein-Gordon equation).

For a massive fermion at rest ($p = 0$), we have

$$p^0 \chi = m\chi \quad \text{and} \quad p^0 \phi = -m\phi.$$  \hspace{1cm} (22)

Positive-energy solutions $\psi^p=0^+$ must therefore have $\phi = 0$ and negative energy solutions $\psi^p=0^-$ have $\chi = 0$, as follows:

$$\psi^p=0^+ = \begin{pmatrix} \chi \\ 0 \end{pmatrix} e^{-imt}, \quad \text{and} \quad \psi^p=0^- = \begin{pmatrix} 0 \\ \phi \end{pmatrix} e^{imt}.$$  \hspace{1cm} (23)
For particles which are not at rest \((p \neq 0)\), the solution is then dictated by eq. (22), with the requirement that it reduces to eq. (24) for \(p = 0\). For positive-energy solutions, we therefore write

\[
\psi_+(x) = \mathcal{N}\left( \frac{\chi_r}{\sqrt{E + m}} \right) e^{-ip \cdot x} \equiv u_r(p) e^{-ip \cdot x}, \quad p^0 = E \equiv \sqrt{p^2 + m^2},
\]

(25)

where \(r = 1, 2\) and \(\mathcal{N}\) is a normalisation conventionally chosen such that \(u^\dagger_r(p)u_r(p) = 2E \delta^{rs}\), which gives \(\mathcal{N} = \sqrt{E + m}\). The spinors \(\chi_1\) and \(\chi_2\) cover the two (spin) degrees of freedom:

\[
\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(26)

Similarly, negative-energy solutions are conventionally written as

\[
\psi_-(x) = \mathcal{N}\left( \frac{\sigma \cdot p}{\sqrt{E + m}} \phi_r \right) e^{ip \cdot x} \equiv v_r(p) e^{ip \cdot x}, \quad p^0 = E,
\]

(27)

with the spinors \(\phi_1\) and \(\phi_2\) again covering the two (spin) degrees of freedom:

\[
\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(28)

The spinors \(u(p)\) and \(v(p)\) therefore represent particle and anti-particle solutions with momentum \(p\) and energy \(E = \sqrt{p^2 + m^2}\).

### 2.2 Spin

Each Dirac spinor has two linearly independent solutions which we stated earlier corresponded to the two possible spin states of a fermion. In this subsection we will define the corresponding spin operator. If we again consider a particle at rest we have

\[
u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.
\]

(29)

These have eigen-values \(\pm \frac{1}{2}\) under the matrix

\[
\frac{1}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & 0 \end{pmatrix},
\]

(30)

One can repeat the same thing for anti-particles and generalise to all the Pauli matrices to deduce the “spin operator”

\[
S = \frac{1}{2} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.
\]

(31)
You can check explicitly that $S^2 = \frac{3}{4} \mathbb{1}_4$, as we would expect. Therefore, for particles at rest, $\mathbf{p} = 0$, the top two components of $\psi_+$ describe fermions with $S_z = +1/2$ (spin up) and $S_z = -1/2$ (spin down) respectively.

In case of a general $\mathbf{p}$ one can consider the projection of the spin-operator along the direction of motion of a particle, i.e. $\mathbf{p}/|\mathbf{p}|$. This gives the helicity operator, $h(\mathbf{p})$

$$h(\mathbf{p}) = \left( \begin{array}{cc} \sigma \cdot \mathbf{p} & 0 \\ 0 & \sigma \cdot \mathbf{p}/|\mathbf{p}| \end{array} \right).$$

This operator satisfies $h(\mathbf{p})^2 = 1$, and hence its eigenvalues are $\pm 1$.

### 2.3 Working with Dirac Spinors

So far we have discussed Dirac spinors, $\psi$, describing spin-1/2 particles and how Dirac used his equation to predict anti-particles. To generate an equation for anti-particles, we first take the Hermitian conjugate of the Dirac equation and find

$$\psi(\mathbf{p}) (-i \gamma^0 \partial_0 + i \gamma^i \partial_i - m) = 0,$$

where the arrows over the derivatives just mean they act on the left, and we have used the fact that $\gamma^{0i} = \gamma^0$ and $\gamma^{ij} = -\gamma^{ji}$. All matrices have to be written on the right because they are multiplying matrices and $\psi^\dagger$ is a row-vector. The above equation does not seem Lorentz covariant. This can be rectified by multiplying the equation by $\gamma^0$ on the right-hand side and using $[\gamma^0, \gamma^i] = 0$. Then we have

$$(\psi^\dagger \gamma^0)(-i \gamma^i \partial_i - m) = 0,$$

or

$$\bar{\psi}(i \partial_0 + m) = 0.$$

The interpretation of the above equation is that the field $\bar{\psi} \equiv \psi^\dagger \gamma^0$ represents an anti-particle.

By construction, the spinors $u(\mathbf{p})$ and $v(\mathbf{p})$ satisfy their respective Dirac equations in momentum space:

$$(\mathbf{p} - m) u(\mathbf{p}) = 0, \quad (\mathbf{p} + m) v(\mathbf{p}) = 0.$$

They also satisfy a number of relations which will prove very useful in calculations of scattering amplitudes. Firstly, they are orthonormal:

$$\overline{u}_r(\mathbf{p}) u_s(\mathbf{p}) = 2m \delta^{rs} = -\overline{v}_r(\mathbf{p}) v_s(\mathbf{p}),$$

$$\overline{u}_r(\mathbf{p}) v_s(\mathbf{p}) = 0 = -\overline{v}_r(\mathbf{p}) u_s(\mathbf{p}).$$

If instead one takes the outer product of spinor and anti-spinor, they also satisfy the following completeness relations:

$$\sum_{r=1}^{2} u_r(\mathbf{p}) \overline{u}_r(\mathbf{p}) = (\mathbf{p} + m) \quad \text{and} \quad \sum_{r=1}^{2} v_r(\mathbf{p}) \overline{v}_r(\mathbf{p}) = (\mathbf{p} - m).$$

These relations can be checked explicitly (see problem sheet).
2.4 Lorentz transformations on spinors

Let us consider the Lorentz transformation of eq. (6). The field \( \psi \) has the transformation property

\[
\psi(x) \rightarrow \psi'(x') = \psi'(\Lambda x) = S(\Lambda)\psi(x) \quad \Rightarrow \quad \bar{\psi}(x) \rightarrow \bar{\psi}'(x') = \bar{\psi}(x)\gamma^0 S^\dagger(\Lambda)\gamma^0 ,
\]

with \( S(\Lambda) \) a suitable \( 4 \times 4 \) matrix. Its explicit form is derived by imposing that the Dirac equation is Lorentz invariant:

\[
(i\partial'_\mu \gamma^\mu - m) \psi'(x') = \left( i(\Lambda^{-1})^\nu_\mu \partial_\nu \gamma^\mu - m \right) S(\Lambda)\psi(x) .
\]

Imposing that \( S(\Lambda) \) satisfies

\[
\gamma^\mu S(\Lambda) = S(\Lambda)\Lambda^\mu_\rho \gamma^\rho ,
\]

we obtain

\[
(i\partial'_\mu \gamma^\mu - m) \psi'(x') = S(\Lambda) \left[ i(\Lambda^{-1})^\nu_\mu \Lambda^\mu_\rho \partial_\nu \gamma^\rho - m \right] \psi(x) = S(\Lambda)(i\partial_\nu \gamma^\nu - m)\psi(x) = 0 ,
\]

so that \( \psi'(x') \) is a solution of the transformed Dirac equation, provided \( \psi(x) \) is a solution of the original one.

Eq. (40) is enough to construct the matrices \( S(\Lambda) \). By direct inspection one observes that

\[
S^\dagger(\Lambda) = \gamma^0 S^{-1}(\Lambda)\gamma^0 \quad \Rightarrow \quad \bar{\psi}'(x') = S^{-1}(\Lambda)\bar{\psi}(x) .
\]

The fact that \( S^{-1}(\Lambda) \neq S^\dagger(\Lambda) \) is not surprising, and is due to the fact that the Lorentz group is non-compact, and therefore it does not admit unitary finite-dimensional representations.

One can construct bi-linear products \( \bar{\psi}\Gamma\psi \), with \( \Gamma \) a \( 4 \times 4 \) matrix. We now show that \( \Gamma \) can be decomposed into a set of bi-linears, each having a definite transformation property under the Lorentz group. Since \( \Gamma \) is \( 4 \times 4 \) matrix, we expect to find 16 such bi-linear products, constructed out of linearly independent matrices. Already we can find 5 such bi-linears:

\[
\bar{\psi}\psi \rightarrow \bar{\psi}S^{-1}(\Lambda)S(\Lambda)\psi = \bar{\psi}\psi \quad \text{ (scalar)} ,
\]

\[
\bar{\psi}\gamma^\mu \psi \rightarrow \bar{\psi}S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi = \Lambda^\mu_\sigma \left( \bar{\psi} \gamma^\nu \psi \right) \quad \text{ (vector)} ,
\]

We can construct 6 more matrices by considering

\[
\Sigma^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] .
\]

Note that \( \gamma^\mu \gamma^\nu \) is not linearly independent from the previous matrices because \( \{\gamma^\mu \gamma^\nu\} = 2g^{\mu\nu}1 \). This gives

\[
\bar{\psi} \Sigma^{\mu\nu} \psi \rightarrow \bar{\psi} S^{-1}(\Lambda) \Sigma^{\mu\nu} S(\Lambda) \psi = \Lambda^\mu_\sigma \Lambda^\nu_\tau \left( \bar{\psi} \Sigma^{\mu\nu} \psi \right) \quad \text{ (tensor)} .
\]
In addition to the four \( \gamma \)-matrices, we can construct their product which is conventionally known as \( \gamma^5 \):

\[
\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix},
\]

which satisfies

\[
(\gamma^5)^2 = 1, \quad \{\gamma^5, \gamma^\mu\} = 0, \quad (\gamma^5)^\dagger = \gamma^5.
\]

The factor of \( i \) is to make the matrix Hermitian. Using \( \gamma^5 \), we can construct 5 more bi-linears

\[
\bar{\psi} \gamma^5 \psi \rightarrow \bar{\psi} S^{-1}(\Lambda) i \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma S(\Lambda) \psi
= \frac{i}{4} \epsilon_{\mu\nu\rho\sigma} \Lambda_\alpha^\mu \Lambda^\nu_\beta \Lambda^\rho_\gamma \Lambda^\sigma_\delta \left( \bar{\psi} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \psi \right)
= \det(\Lambda) \bar{\psi} i \epsilon_{\alpha\beta\gamma\delta} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \psi = \det(\Lambda) \bar{\psi} \gamma^5 \psi \quad \text{(pseudo-scalar)},
\]

\[
\bar{\psi} \gamma^5 \gamma^\mu \psi \rightarrow \det(\Lambda) \Lambda^\mu_\alpha \left( \bar{\psi} \gamma^5 \gamma^\alpha \psi \right) \quad \text{(pseudo-vector)}.
\]

We have then found a set of 16 linearly independent matrices (check that they are linearly independent!)

\[
1, \quad \gamma^5, \quad \gamma^\mu, \quad \gamma^\mu \gamma^5, \quad \Sigma^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu],
\]

so that any bi-linear \( \bar{\psi} \Gamma \psi \) can be written as a sum of terms with definite transformation properties, i.e. transforming in a clear way as a scalar, pseudo-scalar, vector, pseudo-vector and tensor. (This is why the Feynman rule for a pseudo-scalar interacting with a particle-anti-particle pair has a \( \gamma^5 \) for example.)

The most common use of \( \gamma^5 \) is in the projectors \( P_L = (1 - \gamma^5)/2 \) and \( P_R = (1 + \gamma^5)/2 \). You can check explicitly that these behave like projectors (i.e. \( P^2 = P \) and \( P_L P_R = 0 \)). When these act upon a Dirac spinor they project out either the component with “left-handed” chirality or with “right-handed” chirality. These projectors therefore appear when considering weak interactions, for example, as \( W \) bosons only couple to left-handed particles. One has to take care when defining the handedness of antiparticles because

\[
\bar{\psi}_L = \psi_L^\dagger \gamma^0 = \psi_R^\dagger P_L \gamma^0 = \psi_R^\dagger \gamma^0 P_R = \bar{\psi} P_R.
\]

A left-handed anti-particle appears with a right-handed projection operator next to it and vice-versa.

### 3 Quantum Electro-Dynamics

In this section, we will develop the theory of quantum electro-dynamics (QED) which describes the interaction between electrically charged fermions and a vector field (the photon \( A^\mu \)).
3.1 The QED Lagrangian

In this course, we have so far considered spin-0 and spin-1/2 particles. We will postpone a
detailed discussion of spin-1 particles until section 5.1. For the time being, we start from
the Maxwell’s equations in the vacuum in relativistic notation:

$$\partial_\mu F^{\mu\nu} = J^{\nu},$$

where

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu},$$

and $J^{\nu}$ is a conserved current, i.e. satisfying $\partial_\nu J^{\nu} = 0$. Maxwell’s equations can be derived
from the Lagrangian

$$L = L_{\text{em}} + L_{\text{int}},$$

$$L_{\text{em}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

$$L_{\text{int}} = -J^{\mu} A_{\mu},$$

by applying Euler-Lagrange equations

$$\partial_\mu \partial_\nu \frac{\partial L}{\partial \partial_\nu A_\mu} - \frac{\partial L}{\partial A_\mu} = -\partial_\mu F^{\mu\nu} + J^{\nu} = 0.$$ 

The Dirac equation for $\psi$ and its equivalent for $\bar{\psi}$ can be derived from the Lagrangian

$$L_{\text{Dirac}} = \bar{\psi} \left(i \gamma^\mu \partial_\mu - m\right) \psi.$$ 

The starting point for the QED Lagrangian is then the sum of $L_{\text{em}}$ and $L_{\text{Dirac}}$. However, in order to make the theory describe interactions, we must include a term which couples $A^{\mu}$ to $\psi$ and $\bar{\psi}$. If we wish Maxwell’s equation to be valid, this term has to be of the form $L_{\text{int}} = -J^{\mu} A_{\mu}$, with $J^{\mu}$ a conserved vector current. We then observe that the vector current $J^{\mu} = \bar{\psi} \gamma^\mu \psi$ is conserved if $\psi$ is a solution of Dirac equation. In fact

$$\partial_\mu J^{\mu} = \bar{\psi} D_\mu \psi + \bar{\psi} (\partial_\mu \psi) = (-m \bar{\psi} + \bar{\psi} m \psi) = 0.$$ 

Therefore, a good candidate for the electromagnetic current describing an electron of
charge $-e$ is

$$J^{\mu} = -e \bar{\psi} \gamma^\mu \psi,$$

where $-e$ multiplies the vector current so as to be sure that the resulting Coulomb potential
arising from the solution of the static Maxwell’s equations is the expected one. Using the
above current, we obtain:

$$L = L_{\text{em}} + L_{\text{Dirac}} + L_{\text{int}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} \left(i \partial - m\right) \psi + e \bar{\psi} \gamma^\mu \psi A_\mu.$$ 

Notice that $L$ is invariant with respect to the “gauge” transformations

$$\psi(x) \rightarrow \psi'(x) = e^{-ie\alpha(x)} \psi(x), \quad A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x).$$ 

Notice that the addition of the interaction term $L_{\text{int}}$ is equivalent to the replacement

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - ie A_\mu.$$ 

This prescription is known as “minimal coupling” and automatically ensures that the Lagrangian is gauge invariant. The use of gauge invariance to introduce interactions will be covered in detail in the Standard Model course next week. This gives

\[
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\gamma^\mu(\partial_\mu + ieA_\mu))\psi.
\] (60)

The fact that \( \mathcal{L} \) is invariant under the gauge transformations in eq. (62) means that \( A^\mu \) contains unphysical degrees of freedom. This is clear in view of the fact that a massless vector field contains two physical polarisations, whereas \( A^\mu \) has four degrees of freedom. In order to eliminate this degeneracy, a “gauge-fixing” condition is imposed. A possible choice of a gauge condition is the so-called Coulomb gauge, in which \( \nabla \cdot A = 0 \). Although this condition eliminates the two additional degrees of freedom, it breaks Lorentz covariance. A common choice that preserves Lorentz covariance is the Lorentz gauge:

\[
\partial_\mu A^\mu = 0.
\] (61)

This corresponds to choosing the gauge parameter \( \alpha \) such that

\[
\Box \alpha = -\partial_\mu A^\mu
\]

above. In this gauge, the Maxwell equations become \( \Box A^\nu = 0 \).

Notice that the Lorentz gauge condition reduces the number of degrees of freedom in \( A \) from four to three. Even now though \( A^\mu \) is not unique. A transformation of the form

\[
A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \chi, \quad \Box \chi = 0,
\] (62)

will also leave the Lagrangian unchanged. At classical level we can eliminate the extra polarisation “by hand”, but at quantum level this cannot be done without giving up covariant canonical commutation rules. The way out, which can only be summarised, is to add a gauge-fixing Lagrangian \( \mathcal{L}_{gf} \), so that the full QED Lagrangian becomes

\[
\mathcal{L}_{QED} = \mathcal{L}_{\text{em}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{gf}, \quad \mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2.
\] (63)

Using this Lagrangian as a starting point, and an extra condition on physical states, only the two physical polarisations propagate on-shell. Notice that setting \( \xi = 0 \) corresponds to enforcing the Lorentz gauge condition \( \partial_\mu A^\mu = 0 \), otherwise the equations of motions give \( \Box \partial_\mu A^\mu = 0 \), i.e. \( \partial_\mu A^\nu \) is a free field.

### 3.2 Feynman Rules

Feynman developed a method of organising the calculation of scattering amplitudes in terms of diagrams. Starting from a set of vertices (or interactions), each corresponding to a term in the Lagrangian and a set of links (or propagators), you build every possible diagram corresponding to your initial and final state. Each piece comes with a “rule” and the combination of these give the scattering amplitude (actually \( i\mathcal{M} \)).
In the quantum field theory course at this school, you learn how to derive the “Feynman rules” for scalar $\phi^4$ theory. The principles are the same here so in this course we will state the Feynman rules for QED and learn how to work with them. The Feynman rules are shown in figure 4. The left-hand column represents internal parts of the diagram while the right-hand column gives the rules for external fermions and photons.

A few comments are necessary here:

1. Individual pieces of a Feynman diagram are a mixture of matrices, vectors, co-vectors and scalars. They do not commute. The final amplitude is a number and therefore you must follow each fermion line from a spinor (either outgoing particle or incoming anti-particle) through the series of matrices to finish on an anti-spinor (either incoming particle or outgoing anti-particle). This corresponds to working backwards along the fermion line. We will see this in the examples which follow. Similarly, all Lorentz indices corresponding to photons have to be contracted.

2. The photon propagator term has a free parameter $\xi$. This is due to the gauge freedom we discussed in the previous section. It does not represent a physical degree of freedom and therefore any calculation of a physical observable will be independent of $\xi$. We will most commonly work in Feynman gauge $\xi = 1$.

3. The propagators come with factors of $i\varepsilon$ in the denominator, otherwise they would have poles on the real axis and any integral over $p$ would not be well-defined. The
Figure 5: Building the leading-order Feynman diagram for Coulomb scattering. We start from the initial and final states on the left-hand side. The diagram on the right is the only way to connect these with up to two vertices.

A factor of \(i\varepsilon\) prescribes which direction to travel around the poles. This choice corresponds to the “Feynman prescription”, which ensures causality.

4. The interaction vertex contains only one flavour of fermion. We know that the emission of a photon does not change an electron to a quark for example.

5. There are additional factors of \((-1)\) in the following scenarios:
   
   (a) an anti-fermion line runs continuously from an initial to a final state;
   (b) there is a closed fermion loop;
   (c) between diagrams with identical fermions in the final state.

   These arise from the anti-commutation properties of fermionic operators which is beyond the scope of this course. This sign can be important to get the relative phase between diagrams correct, as happens for instance in Bhabha scattering.

Examples: Coulomb Scattering

As a first example, we consider Coulomb scattering:

\[
e(p)\mu(k) \rightarrow e(p')\mu(k').
\]

(64)

We start by drawing the external particles, see left-hand side of fig. 5. We now want to find all possible ways to connect these. There is no direct interaction between an electron and a muon but both interact with a photon, so a possible connected diagram is the one shown on the right-hand side. In fact, this is the only possible diagram with no more than two vertices. The number of vertices is directly related to the powers of the coupling \(e\) and therefore the diagram shown on the right is the leading-order (or tree-level) process.

If we consider \(e(p)\mu(k) \rightarrow e(p')\mu(k')\) or \(e^+(p)\mu^-(k) \rightarrow e^+(p')\mu^-(k')\) instead, there are two diagrams with two vertices, i.e. at \(\mathcal{O}(e^2)\) (try this!). Both have to be added before squaring the amplitude to have the tree-level contribution to the cross section.
If we allow ourselves more than two vertices, there are many more diagrams we can draw. Since the number of external particles doesn’t increase, these must contain closed loops and, therefore, they represent higher-loop processes. In this course, we will limit ourselves to tree-level processes. Loop-diagrams will be covered in the phenomenology course.

Now we will construct the tree-level amplitude for Coulomb scattering from the rules in Fig. 4. Keeping in mind the earlier warning about the ordering of matrices and spinors, we take each fermion line in turn. The electron line gives

$$\bar{u}(p') \ (ie\gamma^\mu) \ u(p). \ (65)$$

In spin-space, this is co-vector–matrix–vector, which is a number. In Lorentz space it has one free index $\mu$ and is therefore a vector. Similarly, for the muon line we get

$$\bar{u}(k') \ (ie\gamma^\nu) \ u(k). \ (66)$$

Lastly, for the propagator with momentum $q = p' - p = k - k'$ in Feynman gauge, we get

$$-ig_{\mu\nu} \frac{q^2}{q^2 + i\varepsilon}, \ (67)$$

so that the full amplitude is

$$i\mathcal{M} = ie^2 \left[ \bar{u}(p') \gamma^\mu u(p) \right] \frac{g_{\mu\nu}}{q^2} \left[ \bar{u}(k') \gamma^\nu u(k) \right]. \ (68)$$

We will drop the $i\varepsilon$ from now on, as we will not need it in this example.

Just as in quantum mechanics, in order to compute the probability of this process happening, we must calculate $|\mathcal{M}|^2$. We will now add specific indices to label the spins, $r, r', s, s'$. In order to describe an unpolarised physical scattering process, we will average over initial-state spins and sum over final-state spins. This convention is represented by a bar as follows:

$$|\mathcal{M}|^2 = \frac{1}{2} \sum_{r=1}^{2} \sum_{s=1}^{2} \sum_{r'=1}^{2} \sum_{s'=1}^{2} |\mathcal{M}|^2$$

$$= \frac{1}{4} \frac{e^4}{(q^2)^2} \sum_{r,r',s,s'} \left[ \bar{u}_{r'}(p') \gamma^\mu u_r(p) \right] \left[ \bar{u}_{r'}(p') \gamma^\rho u_r(p) \right]^*$$

$$\times \sum_{s,s'} \left[ \bar{u}_{s'}(k') \gamma_\mu u_s(k) \right] \left[ \bar{u}_{s'}(k') \gamma_\rho u_s(k) \right]^*, \ (69)$$

where we have explicitly evaluated the metric contractions for brevity.

To evaluate the products in eq. (69) we will use the results from section 2.1. We will take the pieces corresponding to the electron line first. Since $[\bar{u}_{r'}(p') \gamma^\rho u_r(p)]^*$ is a number, its complex conjugate is its hermitian conjugate. Therefore

$$[\bar{u}_{r'}(p') \gamma^\rho u_r(p)]^* = u_r^\dagger(p) \gamma^\rho \gamma^0 u_{r'}(p') = u_r^\dagger(p) \gamma^0 \gamma^\rho u_{r'}(p') = \bar{u}_r(p) \gamma^\rho u_{r'}(p'), \ (70)$$
where we have used $\gamma^\mu = \gamma^0 \gamma^\nu \gamma^0$, which you showed on the problem sheet. We now use eq. (37) to find

$$
\sum_{r,r'} [\bar{u}_{r'}(p') \gamma^\mu u_r(p)] [\bar{u}_{r'}(p') \gamma^\nu u_r(p)]^* = \sum_{r,r'} [\bar{u}_{r'}(p') \gamma^\mu u_r(p) \bar{u}_{r'}(p') \gamma^\nu u_r(p)]^*
= \sum_{r'} \bar{u}_{r'}(p') \gamma^\mu (p + m) \gamma^\nu u_{r'}(p') \tag{71}
$$

We will use $m$ for the electron mass and $M$ for the muon mass. It is now useful to add a component index in spinor-space like you would do in normal linear algebra. Schematically we have

$$
\sum_{r'} \bar{u}_{r'} \Gamma_{ij} u_{r'j}, \tag{72}
$$

where $\Gamma$ represents the chain of $\gamma$-matrices in eq. (71). Now that we are explicitly labelling the components, we can swap the order of the terms to get

$$
\sum_{r'} \Gamma_{ij} u_{r'j} \bar{u}_{r'i} = \Gamma_{ij} (p' + m)_{ji} = \text{Tr}(\gamma^\mu (p + m) \gamma^\nu (p' + m)). \tag{73}
$$

We could have anticipated that we would get a trace as we need to get a single number from a series of matrices. Working from the anti-commutation relations, one can readily show the following identities (see problem sheet):

$$
\text{Tr}(\text{odd number of } \gamma \text{ matrices}) = 0, \quad \text{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}, \quad \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{74}
$$

Therefore, eq. (73) equals

$$
4p_\nu p'_\mu (g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) + 4m^2 g^{\mu\nu}. \tag{75}
$$

The same series of steps gives

$$
\sum_{s,s'} [\bar{u}_{s'}(k')] \gamma_\mu u_s(k)] [\bar{u}_{s'}(k')] \gamma_\rho u_s(k)]^* = 4k_\alpha k'_\beta (g_{\mu\alpha}g_{\rho\beta} - g_{\mu\rho}g_{\alpha\beta} + g_{\mu\beta}g_{\alpha\rho}) + 4M^2 g_{\mu\rho}. \tag{76}
$$

Substituting these results into eq. (69) gives

$$
|\mathcal{M}|^2 = \frac{8e^4}{(2q^2)^2} \left( (pk) (p'k') + (pk')(pk) + 2m^2 M^2 - M^2 (pp') - m^2 (kk') \right) \tag{77}
$$

We will now rewrite the invariants which appear in the above equation in terms of the centre-of-mass energy squared, $s$ and the exchanged momentum-squared, $q^2 = t$. We have

$$
\begin{align*}
2(pk) &= (p + k)^2 - m^2 - M^2 = s - m^2 - M^2, \\
2(p'k') &= (p' + k')^2 - m^2 - M^2 = s - m^2 - M^2, \\
2(pp') &= -(p - p')^2 + 2m^2 = -q^2 + 2m^2, \\
2(kk') &= -q^2 + 2M^2, \\
2(pk') &= 2p \cdot (p + k - p') = s + q^2 - m^2 - M^2, \\
2(p'k) &= s + q^2 - m^2 - M^2.
\end{align*} \tag{78}
$$
which finally gives
\[ |\mathcal{M}|^2 = \frac{2e^4}{(q^2)^2} \left( (s - m^2 - M^2)^2 + (s + q^2 - m^2 - M^2)^2 + 2q^2(m^2 + M^2) \right). \]  
(79)

This expression can be further simplified by introducing the further invariant \( u = (p - k')^2 = (p' - k)^2: \)
\[ |\mathcal{M}|^2 = \frac{2e^4}{t^2} \left( (s - m^2 - M^2)^2 + (u - m^2 - M^2)^2 + 2t(m^2 + M^2) \right). \]  
(80)

The above equation gives the probability that the corresponding process occurs at a given point in phase space. In the next section, we will derive how to calculate a total cross section (or a total decay width) from amplitudes squared.

### 4 Calculation of Cross Sections

Ultimately it is not the amplitude we really want to calculate, but its integral over phase space to give the total cross section if it is a scattering process or the total decay width if it is a decay.

#### 4.1 Phase Space Integrals

We must integrate over all the allowed phase space, which means all possible momentum configurations of the final-state particles. This result, divided by the flux of incoming particles, will give the total cross section.

In principle, we must integrate over a 4-dimensional phase space for each particle \( f \) in the final state, but we must impose that each satisfies its on-shell condition \( p^2_f = m^2_f \).

We therefore must have
\[ \prod_f \int \frac{d^4p_f}{(2\pi)^4} (2\pi)\delta(p^2_f - m^2) \Theta(p^2_f) = \prod_f \int \frac{d^4p_f}{(2\pi)^4} (2\pi) \delta((p^0_f)^2 - (p^2_f - m^2) \Theta(p^0_f)) \]
\[ = \prod_f \int \frac{d^3p_f}{(2\pi)^3(2E_f)}, \]  
(81)

where \( E_f = \sqrt{p^2_f + m^2} \). Although the final expression explicitly separates the dependence on \( E \) and \( \mathbf{p} \), it is still Lorentz invariant as the original expression is clearly Lorentz invariant. Eq. (81) is frequently referred to as the Lorentz Invariant Phase Space measure (LIPS). The factors of \( 2\pi \) correspond to the conventions used for momentum space integrations in QFT.

We now need to normalise this expression to the flux of incoming particles. This is done by multiplying by the flux factor, \( \mathcal{F} \). For the scattering of two incoming particles, this is
usually written as
\[ F = \frac{1}{4E_a E_b |v_a - v_b|}, \quad (82) \]
where \( E_i \) and \( v_i \) are the energy and velocity of each incoming particle.\(^1\) A neater, equivalent form which explicitly demonstrates the Lorentz invariance of this quantity is
\[ F = \frac{1}{4\sqrt{(p_a p_b)^2 - m_a^2 m_b^2}}. \quad (83) \]
In the massless limit \( s \gg m_1, m_2 \), this simplifies to \( F \simeq 1/(2s) \). Finally, we must impose total conservation of momentum to find
\[ \sigma = F \left( \prod_f \int \frac{d^3p_f}{(2\pi)^3(2E_f)} \right) |M|^2 (2\pi)^4 \delta^4 \left( \sum_f p_f - p_1 - p_2 \right). \quad (84) \]
If you wish to calculate a total decay width instead, the expression is very similar. The only difference is that the flux factor becomes
\[ F = \frac{1}{2M}, \quad (85) \]
where \( M \) is the mass of the decaying particle. The total decay width, \( \Gamma \), is therefore given by
\[ \Gamma = \frac{1}{2M} \left( \prod_f \int \frac{d^3p_f}{(2\pi)^3(2E_f)} \right) |M|^2 (2\pi)^4 \delta^4 \left( \sum_f p_f - p_M \right). \quad (86) \]

### 4.2 Return to Coulomb Scattering

We may now calculate the relativistic cross section for Coulomb scattering, using our result from section 3.2. Eq. (84) applied to this example gives
\[ \sigma = F \int \frac{d^3p'}{(2\pi)^3(2E'_p)} \frac{d^3k'}{(2\pi)^3(2E'_k)} |M|^2 (2\pi)^4 \delta^4 (p' + k' - p - k). \quad (87) \]
As this expression is Lorentz invariant, we are free to choose which frame to evaluate it in. This is an extremely powerful tool to evaluate these integrals, as a careful choice can lead to considerable simplifications. We will choose the centre-of-mass frame here so that \( p = -k \). We can easily do the the \( k' \) integration using three of the \( \delta \)-functions to give
\[ \sigma = F \int \frac{d^3p'}{(2\pi)^3} \frac{1}{4E'_p E'_k} |M|^2 (2\pi)\delta \left( E'_p + E'_k - E_p - E_k \right). \quad (88) \]
\(^1\)You can find a motivation for the flux factor in Aitchison and Hey and a more complete derivation in Peskin and Schröder chapter 4.5.
We will proceed by transforming to spherical polar coordinates, 
\[
d^3p' = |p'|^2 dp' |d\Omega,\]
where we have written the solid angle, \( \sin \theta d\theta d\phi \), as \( d\Omega \):
\[
\sigma = \frac{F}{(2\pi)^2} \int d\Omega |p'| \frac{|p'|^2}{4E'_PE'_K} |M|^2 \delta \left( E'_P + E'_K - E_P - E_K \right) .
\]
(89)

We now make the change of variable \(|p'| \rightarrow E = E'_P + E'_K\), which has Jacobian factor
\[
\frac{\partial E}{\partial |p'|} = \frac{E|p'|}{E'_P E'_K}
\]
(90)
to get
\[
\sigma = \frac{F}{(2\pi)^2} \int d\Omega dE \frac{|p'|}{4E} |M|^2 \delta \left( E - \sqrt{s} \right) = \frac{F}{(2\pi)^2} \int d\Omega \frac{|p'|}{4\sqrt{s}} |M|^2,
\]
(91)
where it is understood that \( k' = -p' \) with \(|p'|\) determined from \( E = \sqrt{s} \). The only undefined variables are the angles which remain to be integrated over. We could now substitute the expression for \(|M|^2\) explicitly in terms of these angles but it is actually informative to instead study the differential cross section
\[
\frac{d\sigma}{d\Omega} = \frac{F}{16\pi^2} \frac{|p'|}{\sqrt{s}} |M|^2.
\]
(92)

We will now consider the high energy limit where \( s \gg m_e^2, m_\mu^2 \). In this limit, the three Mandelstam invariants are given by
\[
s = 4p^2, \quad t = -4p^2 \sin^2(\theta/2), \quad u = -4p^2 \cos^2(\theta/2),
\]
(93)
which gives
\[
|M|^2 \approx 2e^4 \frac{s^2 + u^2}{t^2} = \frac{2e^4}{\sin^4(\theta/2)} \left( 1 + \cos^2(\theta/2) \right).
\]
(94)

Note that this amplitude squared has no dependence on the azimuthal angle \( \phi \). Using the conventional notation \( \alpha = e^2/(4\pi) \), we obtain
\[
\frac{d\sigma}{d\Omega} \approx \frac{\alpha^2}{2s} \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)}.
\]
(95)

4.3 The Coulomb Potential

The same calculation may be used to calculate the cross section for the scattering of a relativistic particle from an external Coulomb potential by working in the rest frame of the muon and taking \( m_\mu \rightarrow \infty \). This is illustrated in fig. 6.
Repeating the same calculation in this limit yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4k^2v^2\sin^4(\theta/2)} (1 - v^2\sin^2(\theta/2))$$

(96)

where \( v = |k|/E_k \) and

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \frac{\alpha^2}{4k^2v^2\sin^4(\theta/2)}$$

(97)

is the Rutherford cross section which was calculated in preschool problem 9. The extra \( v^2 \)-term in eq. (96) then gives the relativistic correction to this. This result is entirely due to the electron being a spin-1/2 particle. If it were spin-0 instead, \(|M|^2\) would look much simpler as there are no fermion traces to be performed and in that case we would find that there is no relativistic correction.

### 4.4 \( e^+e^- \) Annihilation

The calculation we have just performed is almost identical to \( e^+(p')e^-(p) \rightarrow \mu^+(k)\mu^-(k') \). Although this now involves anti-particles, there is still one single diagram at leading-order and the trace algebra is very similar. Indeed we can re-interpret the incoming \( e^+ \) as an outgoing \( e^- \) with momentum \(-p'\), and the outgoing \( \mu^+ \) as an incoming \( \mu^- \) with momentum \(-k\). Then we do find explicitly that

$$|\mathcal{M}_{e^+(p')e^-(p)\rightarrow\mu^+(k)\mu^-(k')}|^2 = |\mathcal{M}_{e^-(p)\mu^-(k)\rightarrow\mu^+(k')\mu^-(k')}|^2.$$  

(98)

This is an example of “crossing symmetry”. Note in general that there is an additional minus sign for each fermion which swaps from the initial to final state or vice versa. This is because, for example,

$$\sum_r u_r(p')\bar{u}_r(p') = p' + m \rightarrow \sum_r v_r(-p')\bar{v}_r(-p') = -p' - m = -(p' + m).$$  

(99)
In this case there are two minus signs whose combined effect gives just one. If in $e^+e^-$-annihilation we take the approximation $m_e = 0$, we find

$$|\mathcal{M}|^2 = \frac{8e^4}{s^2} \left[ (pk)^2 + (pk')^2 + m^2_\mu(kk') \right],$$

(100)

Once again, choosing to work in the centre-of-mass frame, we find

$$\left( \frac{d\sigma}{d\Omega} \right)_{e^+e^-\rightarrow\mu^+\mu^-} = \frac{\alpha^2}{4s} \sqrt{1 - \frac{4m^2_\mu}{s}} \left( 1 + \frac{4m^2_\mu}{s} + \left( 1 - \frac{4m^2_\mu}{s} \right) \cos^2 \theta \right).$$

(101)

If we again take the high-energy limit where $s \gg m^2_\mu$, this reduces to

$$\left( \frac{d\sigma}{d\Omega} \right)_{e^+e^-\rightarrow\mu^+\mu^-} = \frac{\alpha^2}{4s} (1 + \cos^2 \theta).$$

(102)

We can now convert the above result to a total cross section by performing the integral over the solid angle. This gives

$$\sigma(e^+e^- \rightarrow \mu^+\mu^-) \simeq \frac{4\pi \alpha^2}{3s}.$$ 

(103)

Now, when an electron and positron annihilate, other fermions may be produced. If these are quarks, they are then observed in the detector as hadrons. The same calculation gives

$$\sigma(e^+e^- \rightarrow \text{hadrons}) = \frac{4\pi \alpha^2}{3s} N_c \sum_{i=1}^{n_f} Q_i^2,$$

(104)

plus higher-order corrections, where there are $N_c$ colours in each of the $n_f$ massless flavours of quarks with charge $Q_i$. Therefore the ratio

$$R = \frac{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}{\sigma(e^+e^- \rightarrow \text{hadrons})}$$

(105)

has been used to measure the number of colours to be $N_c = 3$.

5 Photon Scattering

In this section we will calculate the scattering amplitude for $e\gamma \rightarrow e\gamma$. In order to do that we need first to consider how to treat incoming and outgoing photons.

5.1 Photon Polarisation

We seek to find a plane-wave solution corresponding to a free photon (like our treatment for Dirac particles in section 2.1). It will have the form

$$A^\mu(x) = \varepsilon^\mu(k) e^{-ik\cdot x},$$

(106)
where $\varepsilon^\mu(k)$ is the polarisation vector of the photon. In the Lorentz gauge of eq. (61), the photon equation of motion in eq. (51) is

$$\Box A^\mu = 0 ,$$

and is automatically satisfied by a solution of the form in eq. (106), provided $k^2 = 0$. The Lorentz gauge condition gives an additional constraint on the polarisation vector

$$k \cdot \varepsilon(k) = 0 .$$

However, there is still freedom here because, given a polarisation vector $\varepsilon$ which solves this equation, any other vector of the form $\varepsilon' = \varepsilon + \lambda k$ will also be a solution, which corresponds to the propagation of an extra unphysical longitudinal photon, with a polarisation proportional to $k_\mu$. This freedom is usually used to set $\varepsilon^0 = 0$ such that $k \cdot \varepsilon = 0$ so that the two physical polarisations $\varepsilon^\alpha$, with $\alpha = 1, 2$, are in the transverse direction, and are chosen to be orthonormal. A useful relation we will use in the following is

$$\sum_{\alpha=1}^{2} \varepsilon^{\alpha i}(k) \varepsilon^{\alpha j}(k) = \delta^{ij} - \hat{k}^i \hat{k}^j , \quad \text{where} \quad \hat{k}^i = \frac{k^i}{|k|} = \frac{k^i}{k^0} .$$

The Feynman rule for an incoming photon is simply $\varepsilon^\mu(k)$ while for an outgoing photon it is $\varepsilon^{*\mu}(k)$, as shown in Fig. 4.

As for fermion spins, for unpolarised processes you compute the total cross section by averaging over incoming polarisations and summing over outgoing polarisations. Let us consider the case of a general process with one external incoming photon. The matrix element would have the form

$$i \mathcal{M} = A^\mu \varepsilon_\mu(k) .$$

The left-hand side is a physical quantity, hence it should give the same result for any choice of the gauge. Had we chosen $\varepsilon + \lambda k$ instead, this implies that $A^\mu k_\mu$ has to vanish. This is a “Ward Identity” for QED, and is therefore a test of gauge-invariance.

Squaring the scattering-amplitude over the physical polarisations gives

$$\sum_{\alpha=1}^{2} |A^\mu \varepsilon^{\alpha}_\mu(k)|^2 = \sum_{\alpha=1}^{2} A^\mu A^{*\nu} \varepsilon^{\alpha}_\mu(k) \varepsilon^{*\alpha}_\nu(k)$$

$$= A^i A^j (\delta^{ij} - \hat{k}^i \hat{k}^j) ,$$

using eq. (109). The equation $A^\mu k_\mu = 0$ implies $A^i \hat{k}^i = A^0$ and hence

$$\sum_{\alpha=1}^{2} |A^\mu \varepsilon^{\alpha}_\mu(k)|^2 = A^i A^i - A^0 A^0 = -A^\mu A^{*\nu} g_{\mu\nu} .$$
This could be done for each photon in turn if there were more in the process, and we find the general result that

$$\sum_{\alpha=1}^{2} \varepsilon_\mu^\alpha \varepsilon_\nu^{\alpha} \rightarrow -g_{\mu\nu}.$$  \hspace{1cm} (113)

We have used the → notation of Peskin and Schröder here as the result is not an exact equality in the absence of the rest of the matrix element, but the result is nonetheless true in any practical calculation.

### 5.2 Compton Scattering

There are two diagrams at leading order for this process, shown in Fig. 7. Following the Feynman rules in Fig. 4 and the rules for external photons in the previous subsection, we find that the sum of the two diagrams gives

$$i\mathcal{M} = -ie^2 \varepsilon^{*\mu}(k') \varepsilon^{\nu}(k) \bar{u}(p') \left( \gamma_\mu \frac{p + k + m}{(p + k)^2 - m^2} \gamma_\nu + \gamma_\nu \frac{p - k' + m}{(p - k')^2 - m^2} \gamma_\mu \right) u(p).$$  \hspace{1cm} (114)

You can check explicitly that the above amplitude does indeed satisfy the appropriate QED Ward Identities, i.e. replacing \( \varepsilon^{\nu}(k) \) with \( k^{\nu} \) gives \( \mathcal{M} = 0 \), and similarly when replacing \( \varepsilon^{*\mu}(k') \) with \( k'^{\mu} \) (see tutorial sheet).

We now square the amplitude to get

$$|\mathcal{M}|^2 = \frac{1}{2} \sum_{\gamma_{\text{pol}}} \frac{1}{2} \sum_{\varepsilon_{\text{spin}}} |\mathcal{M}|^2$$

$$= 2e^4 \left( \frac{(pk)}{(pk')} + \frac{(pk')}{(pk)} + 2m^2 \left( \frac{1}{(pk)} - \frac{1}{(pk')} \right) + m^4 \left( \frac{1}{(pk)} - \frac{1}{(pk')} \right)^2 \right).$$  \hspace{1cm} (115)

The calculation of the spin traces in this case requires the identities

$$\gamma_\mu \gamma^\mu = 4, \quad \gamma_\mu \gamma^\rho \gamma^\mu = -2\gamma^\rho,$$  \hspace{1cm} (116)
Figure 8: The Compton scattering process in the rest frame of the incoming electron.

from the problem sheet. We will again choose a suitable reference frame to simplify the calculation. In this case, it is convenient to work in the rest frame of the incoming electron as shown in fig. 8. We can use energy conservation to compute \( \omega' \):

\[
m^2 = p'^2 = (p + k - k')^2 = m^2 + 2m(\omega - \omega') - 2\omega\omega'(1 - \cos \theta)
\]

\[
\Rightarrow \quad \omega' = \frac{\omega}{1 + (\omega/m)(1 - \cos \theta)} .
\]

In this frame, we therefore have

\[
|\mathcal{M}|^2 = 2e^4 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) .
\]

The explicit dependence on the electron mass cancels with the factors of \( m \) in \( \omega' \). It is however present in the flux factor \( \mathcal{F} = 1/(4m\omega) \). We now compute the integral over the phase space to get

\[
\sigma = \frac{1}{4m\omega} \int \frac{d^3p'}{(2\pi)^3(2E')} \frac{d^3k'}{(2\pi)^3(2\omega')} 2e^4 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) (2\pi)^4 \delta^4 (p' + k' - p - k) .
\]

We can again do the integral over \( d^3k' \) using the spatial parts of the \( \delta \)-function. Then we transfer to spherical polars and find

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) .
\]

A nice check of this result is to take the low-energy limit where \( \omega \ll m \). Then \( \omega \approx \omega' \) and we find

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2} (1 + \cos^2 \theta) .
\]

This is the Thomson cross section for the scattering of classical electromagnetic radiation by a free electron. In the other limit, the high-energy limit where \( \omega \gg m \), we have

\[
\omega' \approx \frac{m}{1 - \cos \theta} \quad \Rightarrow \quad \frac{d\sigma}{d\Omega} \approx \frac{\alpha^2}{2m\omega} \frac{1}{1 - \cos \theta} .
\]
and the cross section is strongly peaked for small angles. This leads to a logarithmic enhancement when you perform the angular integration. These “collinear” logarithms arise whenever massless particles are emitted; this will be discussed in more detail in the phenomenology course.

Note that, since $\omega > \omega'$, eq. (122) holds strictly for $(1 - \cos \theta) > m/\omega$. For smaller angles, eq. (121) holds and

$$\frac{d\sigma}{d\Omega} \simeq \frac{\alpha^2}{m^2}.$$  \hspace{1cm} (123)

The forward (small scattering angle) Compton scattering cross section is then a valuable method to measure the QED coupling $\alpha$.

### 6 Strong Interactions

In this section we will develop the theory of the strong interactions, quantum chromodynamics (QCD). The major difference between QED and QCD is that the gluons are self-interacting because they also carry colour charge (unlike the charge-neutral photon).

#### 6.1 QCD Lagrangian

The particles which carry colour charge are

**Spin-1/2:** six families of quarks (up, charge and top with electric charge +2/3; down, strange and bottom with electric charge -1/3)

For each flavour, there are $N_c = 3$ of these.

**Spin-1:** $8 = (N_c^2 - 1)$ massless gluons.

The QCD Lagrangian for a quark of mass $m$ is

$$L_{QCD} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi}_i (i\slashed{D}_{ij} - m\delta_{ij}) \psi_j,$$

with $D_{ij}^\mu = \partial^\mu \delta_{ij} + ig s t_{ij}^a A^{a\mu}$, $F_{\mu\nu}^a = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a + g_s f^{abc} A_{\mu}^b A_{\nu}^c$.  \hspace{1cm} (124)

The $a$, $i$ and $j$ indices are gauge group indices which are discussed further below. The sum over these is implicit in eq. (124). Each $t^a$ is a $3 \times 3$ matrix in colour space. The $t^a$ matrices do not commute with each other, but obey the following algebra

$$[t^a, t^b] = if^{abc} t^c,$$  \hspace{1cm} (125)

which is reminiscent of the algebra of angular momentum operators, $[J_i, J_j] = i\varepsilon_{ijk} J_k$. Here, in place of the alternating tensor $\varepsilon_{ijk}$, we have the “structure constants” $f^{abc}$ (which also appear in $F_{\mu\nu}^a$). These are also completely anti-symmetric under the swapping of any pair of indices.
Just as the $J_i$ generate the rotation group, $SU(2)$, the $t^a$ generate the colour symmetry group, $SU(3)$. We choose to take the Pauli matrices as a representation of $SU(2)$. For $SU(3)$ we choose to take the representation where $t^a = \frac{1}{2} \lambda^a$ and the $\lambda^a$ are the Gell-Mann matrices:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

In practice, we are not interested in calculating one particular colour component and instead work with sums over all colours which ultimately leads to traces over the $t^a$-matrices. We will see explicit examples of this in the sections that follow and here just collect some useful identities:

$$\text{Tr}(t^a) = 0, \quad \text{Tr}(t^at^b) = \frac{1}{2} \delta^{ab}, \quad \sum_a t^a_{ij} t^a_{jk} = C_F \delta_{ik}, \quad \sum_{a,b} f^{abc} f^{abd} = C_A \delta^{cd},$$

where $C_F = \frac{4}{3}, \quad C_A = 3$.

Notice that we have labelled with $a = 1, \ldots, 8$ the gluon indices and with $i = 1, \ldots, 3$ the quark indices. Particular care must be taken when these identities combine to give a trace of a $\delta$-function:

$$\delta^{aa} = N_c^2 - 1 = 8 \quad (\text{the number of gluons})$$

$$\delta_{ii} = N_c = 3 \quad (\text{the number of quarks}).$$

The QCD Lagrangian $\mathcal{L}_{\text{QCD}}$ is invariant under the infinitesimal “gauge” transformations

$$\psi_i(x) \rightarrow (\delta_{ij} - i g_s \theta^a(x) t^a_{ij} ) \psi(x),$$

$$A^a_{\mu}(x) \rightarrow A^a_{\mu}(x) + D^a_{\mu} \theta^b(x),$$

where $D^a_{\mu}$ is the covariant derivative in the “adjoint” representation, the one under which the gluon fields transforms under $SU(3)$, as opposed to the “fundamental” representation, which rules the transformation of quark fields. In particular, the adjoint covariant derivative is given by

$$D^a_{\mu} = \partial_{\mu} \delta^{ab} + ig_s A^c_{\mu} (T^c)^{ab}, \quad (T^c)^{ab} = i f^{abc} = -i f^{abc}.$$

The matrices $T^a$, as needed for any generator of a representation of $SU(3)$, satisfy the same commutation rules as $t^a$:

$$[T^a, T^b] = if^{abc} T^c.$$
These are nothing else than the Jacobi identity satisfied by the structure constants $f^{abc}$:

$$f^{abc} f^{cde} + f^{bde} f^{cae} + f^{dac} f^{ebe} = 0.$$  \hspace{1cm} (132)

Notice that the gauge transformation for $A_\mu^a$ involves the strong coupling $g_s$:

$$A_\mu^a \rightarrow A_\mu^a + \partial_\mu \theta^a + g_s f^{abc} \theta^b A_\mu^c = A_\mu^a + \partial_\mu \theta^a + O(g_s),$$  \hspace{1cm} (133)

and only at lowest order in $g_s$ does it reduce to the analogous transformation for QED.

As in QED, in order to quantise the QCD Lagrangian, we need to introduce a “gauge-fixing” term, for instance

$$L_{gf} = -\frac{1}{2\xi} (\partial_\mu A_\mu^a)^2.$$  \hspace{1cm} (134)

We now describe the Feynman rules for QCD. The quark and gluon propagators are identical to those for QED except they are also accompanied by the appropriate delta-function in colour space (see fig. 9). The coupling between two quarks and a gluon is now given in terms of colour matrices, $t^a_{ij}$ as shown in fig. 10. Notice that the Dirac matrix $\gamma^\mu$ also still appears as it must for spin-1/2 particles. The colour matrices and Dirac matrices do not interact with each other (they act on different vector spaces). The ‘$a$’ is the “adjoint” index and is associated with the gluon. The $k$ and $j$ are “fundamental” indices associated with the outgoing and incoming fermion line respectively.

Figure 9: The Feynman rules for the quark and gluon propagators (the latter is in Feynman gauge $\xi = 1$).

given in terms of colour matrices, $t^a_{ij}$ as shown in fig. 10. Notice that the Dirac matrix $\gamma^\mu$ also still appears as it must for spin-1/2 particles. The colour matrices and Dirac matrices do not interact with each other (they act on different vector spaces). The ‘$a$’ is the “adjoint” index and is associated with the gluon. The $k$ and $j$ are “fundamental” indices associated with the outgoing and incoming fermion line respectively.

Figure 10: The Feynman rule for a quark-quark-gluon vertex.
Figure 11: Three and four gluon vertices which arise from eq. (124). All momenta are taken to be incoming.

Returning to the Lagrangian, in QCD $F_{\mu\nu}^a$ has an extra term compared to QED, as required by gauge invariance. (Technically this term is present for QED too, but QED is an “Abelian” gauge theory which means that the structure constants are zero). Multiplying out $F_{\mu\nu}^a F_{\mu\nu}^a$ give extra terms with 3 and 4 gauge fields. These correspond to new three- and four-gluon vertices as shown in fig. 11.

6.2 Gauge Invariance

The presence of the non-commuting colour matrices illustrates that $SU(3)$ is a non-Abelian gauge group. We can see the effect of this by studying the QCD equivalent of photon pair production, $q(p) \bar{q}(p') \rightarrow g(k) g(k')$, shown in fig. 12. In QED, the matrix element squared

for this process can be obtained from that of Compton scattering via crossing.

One immediate effect is obvious – there is now a third diagram including the three-gluon
vertex. If we sum the contributions from the first two diagrams we find
\[ \mathcal{M}^{(a)+(b)} = A_{\mu\nu}^{(a)+(b)} \epsilon^\mu(k) \epsilon^{\nu}(k'), \]
\[ A_{\mu\nu}^{(a)+(b)} = -ig_s^2 \mathcal{V}(p') \left( \frac{p - k'}{(p - k)^2} \gamma_\mu t^a + \gamma_\mu t^a \frac{p - k'}{(p - k)^2} \gamma_\nu t^b \right) u(p), \] (135)
where we have implicitly assumed that gluon \( k \) has colour \( a \) and polarisation index \( \mu \), and gluon \( k' \) has colour \( b \) and polarisation index \( \nu \). At this order, see eq. (133), gauge invariance corresponds to testing whether the replacement \( \epsilon_\mu \to \epsilon_\mu + \lambda k_\mu \) leaves the amplitude invariant. This is equivalent to testing the condition for the Ward Identity, \( A_{\mu\nu}^{(a)+(b)} k^\mu = 0 \):
\[ A_{\mu\nu}^{(a)+(b)} k^\mu = -ig_s^2 [t^a, t^b] \mathcal{V}(p') \gamma_\nu u(p) \neq 0. \] (136)
The non-zero commutator makes these diagrams alone not gauge-invariant. Adding diagram \((c)\) gives a contribution which exactly cancels this (try this!) but yields another term proportional to \( k'_\mu \). This vanishes when we remember the whole expression is contracted with \( \epsilon'^{\nu}(k') \), and so gauge invariance is only obeyed once we project onto physical polarisations. This wasn’t necessary in QED.

Recall in the QED case in section 3.2, we used \( A_{\mu\nu} k^\mu = 0 \) to show that, in practical calculations, we can always make the replacement
\[ \sum_{\alpha=1}^2 \epsilon^\alpha_\mu \epsilon'^\alpha_\nu \to -g_{\mu\nu}. \] (137)
Although the right-hand summed all polarisations and not only the physical transverse ones, in actual calculations the unphysical longitudinal gluon polarisations automatically cancelled. This is no longer the case in QCD, where one has to sum strictly over physical polarisations. However, this can make calculations more cumbersome, so it might still be useful to sum over all polarisations, and to cancel in some way the unphysical degrees of freedom. How this cancellation is performed depends on the gauge. In covariant gauges, like the Feynman gauge, this is done by introducing extra fields, called the ghost fields. The alternative is to use the so-called physical gauges, that ensure that that only physical degrees of freedom propagate on shell.

**Ghost Fields**

To understand how the cancellation of unphysical polarisations actually arises in a covariant gauge, we need to revert to the case of photon pair production in QED. When we make the replacement in eq. (137), we are exploiting the fact that QED is unitary, i.e. probability is conserved through time evolution. A non-trivial implication of unitarity is that, at the lowest order in perturbation theory, twice the imaginary part of the forward amplitude for the process \( e^+e^- \to e^+e^- \) has to be equal to amplitude squared for the process \( e^+e^- \to \gamma\gamma \), when we integrate over the photon phase space and sum over physical photon polarisations. This is illustrated in Fig. 13, which shows the only intermediate
states that, at the considered order in perturbation theory, give a non-zero imaginary part, namely two virtual photons. Furthermore, it is possible to show that the imaginary part of any Feynman diagram is obtained by putting on shell in all possible ways the intermediate propagators (i.e. cutting the diagram) by replacing \( i/(p^2 - m^2 + i\epsilon) \) in each of them by \( (2\pi)\Theta(p_0)\delta(p^2 - m^2) \). This divides each Feynman diagram into two subdiagrams on either side of the cut. On one side of the cut, one uses standard Feynman rules. On the other sides, one needs to apply complex conjugation to all Feynman vertices and propagators. Cuts of a diagram that conflict with energy-momentum conservation do not give any contribution to the imaginary part. The result of this cutting procedure for the present case is illustrated in Fig. 14. The dashed line on the right-hand side of the figure represents the only cut of the diagram that gives a non-zero imaginary part, obtained by putting on shell the intermediate photon propagators. If the amplitude is computed in the Feynman gauge, for an intermediate photon of momentum \( k \), we have to perform the replacement

\[
-\frac{g_{\mu\nu}}{k^2 + i\epsilon} \rightarrow -g_{\mu\nu}(2\pi)\Theta(k_0)\delta(k^2).
\]

Let us call \( A^{\mu\nu} \) the contribution to the diagram on the left of the cut in Fig. 14. From the Ward identity \( k_\mu A^{\mu\nu} = 0 \), we obtain that the contribution of gluon \( k \) to the imaginary part of the amplitude becomes

\[
\int \frac{d^4k}{(2\pi)^4} (-g_{\mu\nu})(2\pi)\Theta(k_0)\delta(k^2)A^{\mu\nu} = \int \frac{d^3k}{(2\pi)^32k_0} A^{\mu\nu} \sum_{a=1}^{2} \epsilon_\alpha^a(k)\epsilon_{\mu\alpha}^a(k),
\]
where $\alpha = 1, 2$ is the index labelling photon physical polarisations. This verifies explicitly the unitary relation represented in Fig. 13. The latter means that, in QED, making the replacement in eq. (137) corresponds to exploiting the unitarity of the theory to compute an amplitude squared through the imaginary part of the corresponding forward amplitude.

In the case of QCD, as we have seen in the previous section, the fact that $k_\mu A^{\mu} \neq 0$ implies that the amplitude squared for the process $q\bar{q} \to gg$ is not given by the imaginary part of the forward amplitude $q\bar{q} \to q\bar{q}$, when only gluons are considered as intermediate states. In fact, the cut forward amplitude contains the contribution of non-physical longitudinal polarisations, which do not contribute to the amplitude squared for $q\bar{q} \to gg$. This would violate unitarity, so there has to be additional fields that are responsible for the cancellation of the contribution of non-physical polarisations in the imaginary part of the forward amplitude. These new fields are called ghosts. They are scalar fields, but satisfy Pauli exclusion principle like fermions. They transform under $SU(3)$ in the same way as gluons, i.e. in the adjoint representation. The Feynman rules for ghosts are shown in fig. 15. They can propagate and couple to gluons, but never appear in physical final states. If

$$f^{\mu \nu \rho} g_{\nu \rho} \delta^{ab} \frac{i}{q^2 + i\varepsilon}$$

Figure 15: The Feynman rules for ghost fields, which are constructed explicitly to cancel unphysical degrees of freedom.

we now consider the imaginary part of the forward $q\bar{q}$ amplitude, at the lowest order in perturbation theory we need to include not only gluons as intermediate states, but ghosts as well, as pictorially illustrated in Fig. 16. The ghost-antighost loop contributes to the imaginary part of the forward amplitude with a factor $(-1)$, just like a normal fermion loop, so as to cancel the contribution of the unphysical longitudinal gluon polarisations when summing over all diagrams. The resulting imaginary part equals the amplitude squared for the process $q\bar{q} \to gg$, integrated over the gluon phase space and summed over physical gluon polarisations, as required by unitarity of QCD.

**Physical Gauges**

Alternatively, we can impose a so-called “physical gauge” condition on the gluon fields to eliminate unphysical polarisations from the start. This eliminates the need for ghosts, which do not interact with gluons anymore, but complicates the gluon propagator. In place of the Lorentz gauge condition $\partial^\mu A^{\mu} = 0$, we impose

$$A^{a}_{\mu} n^\mu = 0 ,$$

(140)
Figure 16: Pictorial representation of the unitarity constraint for QCD discussed in the text. Longitudinal polarisations for on-shell gluons in the cut amplitude on the left-hand side of the equality are cancelled by the contribution of the ghost-antighost loop. Each blob represents the sum of the three diagrams in Fig. 12.

for some arbitrary reference vector $n^\mu$. This is done by adding the gauge-fixing Lagrangian

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(A^\mu_n n^\mu)^2,$$  \hspace{1cm} (141)

and taking the limit $\xi \to 0$, thus enforcing the gauge condition in eq. (140).

The new expression for the propagator (for $\xi = 0$) is shown in fig. 17. When we use a physical gauge, whenever we sum over polarisations, we can make the replacement

$$\sum_{\alpha=1}^2 \epsilon_\mu^\alpha(q) \epsilon^{*\alpha}_\nu(q) \to -g_{\mu\nu} + \frac{q^\mu n^\nu + q^\nu n^\mu}{(qn)} - n^2 \frac{q^\mu q^\nu}{(qn)^2}.$$  \hspace{1cm} (142)

The different choices of reference vector $n^\mu$ correspond to different choices of the gauge. One can explicitly check that results for physical quantities, such as cross sections, are independent of this choice.

A relevant example of a physical gauge is the light-cone gauge, in which $n^2 = 0$. In such a gauge, if we have an on-shell gluon $q = (\omega, q)$, we can choose $n = (1, -q/\omega)$. In this case

$$-g_{\mu\nu} + \frac{q^\mu n^\nu + q^\nu n^\mu}{(qn)} = \sum_{\alpha=1}^2 \epsilon_\mu^\alpha(q) \epsilon^{*\alpha}_\nu(q),$$  \hspace{1cm} (143)

so that the replacement gives exactly the sum over the physical polarisations introduced in section 5.1. The expression in eq. (143) is the one that must be used in covariant gauges if one does not want to introduce unphysical amplitudes squared with ghosts in the final state.
Figure 18: Sample “loop” Feynman diagrams: (a) one of the one-loop corrections to Coulomb scattering and (b) a one-loop correction to the photon propagator.

7 Renormalisation

7.1 Dimensional regularisation and renormalisation scale

As mentioned in section 3.2, starting from the Feynman rules one can construct diagrams with loops, as for example the diagrams shown in fig. 18. The presence of loops means that momentum-conservation at each interaction vertex is no longer sufficient to determine the momentum in each leg. For example, \( k \) can take any value in the diagrams shown in fig. 18. We must therefore integrate over all possible values of unconstrained loop momenta. For example, the result for the diagram in fig. 18(b) is

\[
(ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\text{Tr}[\gamma_\mu(k + p + m)\gamma_\nu(k + m)]}{(k^2 - m^2)((k + p)^2 - m^2)},
\]

(144)

with \( p \) the photon momentum and \( d \) the number of space-time dimensions. As the integral runs over all values of \( k \), it includes very large values of \( k \). Counting the powers of \( k \), there are six of them in the numerator and four in the denominator, which implies that this integral diverges. In general, for any integral of the form

\[
\int \frac{d^d k}{(2\pi)^d} \frac{N(k)}{M(k)}
\]

(145)

we define the superficial degree of divergence, \( D \), to be the result of the naïve power-counting:

\[
D = d + \text{(powers of } k \text{ in } N) - \text{(powers of } k \text{ in } M) \tag{146}
\]

If \( D \geq 0 \), then the integral is said to be superficially divergent. Such divergences are called ultra-violet (UV) because they arise whenever loop momenta get large. The boundary case of \( D = 0 \) is a logarithmic divergence (think of \( \int dk \, 1/k \)). The term “superficial” is used because there can be other factors which can affect the actual degree of divergence. In the example above, gauge invariance actually implies that the final result of the integral in eq. (144) must be proportional to \( (p^2 g_{\mu\nu} - p_\mu p_\nu) \). Therefore the divergence is only logarithmic, and not quadratic as it appears from naive power counting.

The main point, though, is not the degree of divergence, but the fact that one finds divergences at all. These higher-loop corrections were supposed to be corrections in the
perturbative series, hence smaller than those appearing at the previous perturbative order. For many years, this caused a major problem for the development of perturbation theory. However, there exists a well-defined procedure to “remove” these divergences which is called renormalisation. The basic idea behind renormalisation is that the parameters appearing in the Lagrangian do not need to be physical quantities, but their value is determined by comparing perturbative predictions to actual experimental data. For instance, the value of $e$ can be extracted by measuring the Compton differential cross section at small angles. Therefore, infinities that eventually appear in perturbative calculations can be in principle reabsorbed in a redefinition of the parameters entering the Lagrangian. In practice, this amounts to rescaling all quantities in the Lagrangian by a “renormalisation constant”, $Z$. For instance, for a field $\phi$ we have
\[ \phi \rightarrow \phi_0 = Z_\phi \phi_R. \] (147)
The field $\phi_0$ is called “bare” field, as opposed to the “renormalised” field $\phi_R$, and $Z_\phi$ is called renormalisation constant. This procedure has to be repeated for all fields, masses and coupling constants. Provided that all infinities in the theory can be removed with a finite number of renormalisation constants $Z$, then the theory is said to be renormalisable. After the renormalisation constants have been fixed, we can calculate all physical quantities in terms of the renormalised quantities and the results will be both finite and unambiguously defined.

The renormalisation constants are calculated according to some procedure that is called “renormalisation scheme”. This consists in computing a suitable set of correlation functions, and imposing that these functions are finite at any order in perturbation theory. In this procedure one finds divergent integrals, which have to be regularised in some way. The regularisation actually provide means to parameterise the divergence. One approach is to implement a momentum cut-off, $\Lambda$, so as to artificially remove the region with large momentum. The most common approach though is called “dimensional regularisation”.

Here we decrease the term $d$ in eq. (146) to a lower value, so that we calculate all integrals in $d = 4 - 2\epsilon$ dimensions instead of $d = 4$. The integration measure becomes
\[ \frac{d^4k}{(2\pi)^4} \rightarrow \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}}, \] (148)
and for each dimensionsless coupling $g_R$ one performs the replacement
\[ g_R \rightarrow \mu^{\frac{d-4}{2\epsilon}} g_R(\mu) = \mu^\epsilon g_R(\mu). \] (149)
The factor of $\mu^\epsilon$ is essential to preserve the correct dimensions of the bare coupling in $d$ dimensions. The renormalised coupling $g_R$ stays dimensionsless and depends now on the scale $\mu$. The latter quantity is the famous renormalisation scale and it is the price that we pay for renormalisation as our finite calculations are now all dependent upon $\mu$.

To summarise, the steps to perform renormalisation within dimensional regularisation are:

1. Compute all integrals in terms of renormalised quantities.
2. All UV divergences appear as $1/\epsilon$-poles.

3. Define the renormalisation functions $Z$ so as to cancel the poles in $\epsilon$ (and maybe some finite terms).

After renormalisation, eq. (147) depends on both $\epsilon$ and $\mu$, as follows:

$$\phi_0(\epsilon) = Z_\phi(\mu, \epsilon) \phi_R(\mu),$$

and a similar expression holds for all couplings and masses. Both $\phi_0$ and $Z$ are infinite for $\epsilon \to 0$, whereas $\phi_R(\mu)$ stays finite, but depends on the unphysical renormalisation scale $\mu$.

In a renormalised theory then, even tree-level diagrams depend on the renormalisation scale, through the coupling for example. The dependence on the renormalisation scale would disappear only if we were able to calculate physical quantities to all orders in perturbation theory. Although this is unpractical, calculating one or two extra orders in perturbation theory can reduce the dependence considerably. However, this does mean that any theoretical calculation now depends on a free parameter, and it is exactly this parameter which leads to a way to estimate the “theory uncertainty”. In fact, consider an observable $O(\alpha_R(\mu), \mu, \{Q_i\})$, where $\{Q_i\}$ is a set of characteristic scales for the process. If we know $O^{(n)}$, the perturbative expansion of $O$ at order $n$ in perturbation theory, we have

$$O^{(n)}(\alpha_R(\mu'), \mu', \{Q_i\}) = O^{(n)}(\alpha_R(\mu), \mu, \{Q_i\}) + O(\alpha_R^{n+1}(\mu)), \quad (151)$$

so that the variation of $\mu$ around some central value $\mu_0$ produces automatically a higher-order term. Notice that $O^{(n)}(\alpha_R(\mu), \mu, \{Q_i\})$ might contain $\ln(Q_i/\mu)$. This is why the central scale $\mu_0$ is normally chosen of the order of the typical value that the scales $Q_i$ can assume. For example in $gg \to H$, one would typically take $\mu_0 \sim m_H$.

The obvious way to gauge how the strength of the dependence on the scale in a calculation is to vary the scale and see how the result varies. If the dependence is very weak, the result will be negligible. If the dependence is very strong, the variation will be large. The consensus of the community is to quote the theoretical uncertainty when the central scale is varied by a factor of 2 in each direction. One should remember that this is only an uncertainty of the dependence on the renormalisation scale and not a strict error bar. This is illustrated by the plot in fig. 19, which is taken from Gehrmann-De Ridder, Gehrmann, Glover & Pires, arXiv:1301.7310. It shows the scale dependence for inclusive jet production in the gluon-gluon channel at LO, NLO and NNLO. Indeed the variation decreases each time indicating that the sensitivity to the scale is decreasing. The fact that the lines do not overlap is a clear sign that these uncertainty bands are not error bands.

### 7.2 Running Coupling

Suppose we have chosen a renormalisation scale $\mu$. How do we measure a coupling $\alpha_R(\mu)$? We normally consider an observable $O(\alpha_R(\mu'), \mu', \{Q_i\})$, compute it at the highest possible
order in perturbation theory, and compare the obtained number with experimental data:

\[ O^{(n)}(\alpha_R(\mu'), \mu', \{Q_i\}) = O_{\exp} \Rightarrow \alpha_R(\mu) . \]  

By doing this for various observables, characterised by different typical scales \( \mu \), one can actually measure the dependence of the coupling on the renormalisation scale \( \mu \). This dependence can be predicted theoretically, and the comparison of the predicted dependence with the one that is actually observed represents one of the most stringent tests of the validity of a given QFT. This is illustrated for QCD in fig. 20, where one sees an astonishing agreement between the predicted “running” of the QCD coupling with the renormalisation scale \( Q \), and what is observed in experimental data.

Figure 19: Plot showing the scale dependence for inclusive jet production at LO, NLO and NNLO, taken from Gehrmann-De Ridder, Gehrmann, Glover & Pires, arXiv:1301.7310.

The theoretical object that dictates how a coupling evolves with the renormalisation scale is the beta function \( \beta(\alpha_R) \), defined as

\[ \mu^2 \frac{\partial \alpha_R}{\partial \mu^2} = \beta(\alpha_R) = -\beta_0 \alpha_R^2 - \beta_1 \alpha_R^3 + \ldots . \]  

There are various ways to compute the beta function, which in general depends on the renormalisation scheme used. However, one can show that the first two coefficient of the beta function, \( \beta_0 \) and \( \beta_1 \), are independent of the renormalisation scheme. If we consider a scheme tied to dimensional regularisation (e.g. the so-called MS scheme), one has the relation

\[ \alpha_0(\epsilon) = \mu^{2\epsilon} Z_g^2(\epsilon, \mu^2) \alpha_R(\mu^2) , \]  

where \( \alpha_0 = g_0^2/(4\pi) \) and \( \alpha_R = g_R^2/(4\pi) \). The crucial observation is that the bare coupling \( \alpha_0 \) does not depend on \( \mu \). Therefore, its logarithmic derivative with respect to \( \mu^2 \) is zero:

\[ 0 = \mu^2 \frac{\partial \alpha_0}{\partial \mu^2} = \mu^{2\epsilon} Z_g^2(\epsilon, \mu^2) \left[ \left( \epsilon + \frac{\mu^2}{Z_g^2(\mu^2)} \right) \frac{\partial Z_g^2}{\partial \mu^2} \right] \alpha_R + \mu^2 \frac{\partial \alpha_R}{\partial \mu^2} \] .
This gives
\[ \mu^2 \frac{\partial \alpha_R}{\partial \mu^2} = -\left( \epsilon + \frac{\mu^2}{Z_g^2} \frac{\partial Z_g^2}{\partial \mu^2} \right) \alpha_R \equiv \beta(\epsilon, \alpha_R) \rightarrow \beta(\alpha_R) = -\lim_{\epsilon \to 0} \frac{\mu^2}{Z_g^2} \frac{\partial Z_g^2}{\partial \mu^2} \alpha_R, \quad \epsilon \to 0. \]  
(156)

In any scheme based on dimensional regularisation we have
\[ Z_g(\epsilon, \mu^2) = 1 + \frac{\alpha_R(\mu^2)}{\epsilon} Z_g^{(1)} + \ldots. \]  
(157)

Therefore the first term of the beta function is just obtained from the $1/\epsilon$ pole of $Z_g$, as follows
\[ \beta(\alpha_R) = -\lim_{\epsilon \to 0} \frac{\mu^2}{Z_g^2} \frac{\partial Z_g^2}{\partial \mu^2} \alpha_R = -\lim_{\epsilon \to 0} \frac{2Z_g^{(1)}}{\epsilon} = -\beta_0 \alpha_R^2 \Rightarrow \beta_0 = -2Z_g^{(1)}. \]  
(158)

The calculation of $Z_g^{(1)}$ can be performed using any quantity that involves an interaction vertex. A way that is common to both QED and QCD is to consider the renormalised interaction Lagrangian
\[ \mathcal{L}_{\text{int}} \to Z_g Z_2 \sqrt{Z_3} (g_R \bar{\psi}_R \mathcal{A}_R \psi_R) = Z_1 (g_R \bar{\psi}_R \mathcal{A}_R \psi_R), \quad Z_g = \frac{Z_1}{Z_2 \sqrt{Z_3}}. \]  
(159)

Here we have used the ubiquitous notation $Z_\psi = \sqrt{Z_2}$ and $Z_A = \sqrt{Z_3}$. The function $Z_1$ contains all UV divergences associated with loop corrections to the interaction vertex, whereas $Z_2$ and $Z_3$ contain UV divergences arising in the calculations of the fermion
and gauge-boson propagators respectively. In QED, a powerful Ward identity implies $Z_1 = Z_2$, so that the beta function can be calculated just from all the loop corrections to the propagator in the unrenormalised theory. For the case of QED

$$\beta_0 = -2Z_g^{(1)} = Z_3^{(1)} = -\frac{1}{3\pi}. \quad (160)$$

Inserting this expression in the beta function we obtain

$$\beta_{\text{QED}}(\alpha) = \frac{1}{3\pi} \alpha^2. \quad (161)$$

which means that the QED coupling, at least until the beta function is dominated by its first term, becomes stronger with energy.

In QCD instead the Ward identity $Z_1 = Z_2$ does not hold any more. However, it holds at least for the part of these renormalisation functions that depends on $C_F$. Since, at one loop, $Z_2^{(1)}$ is proportional to $C_F$, its contribution to the beta function cancels exactly with the abelian contribution to $Z_1^{(1)}$. Therefore, the only contributions to the QCD beta function at one loop come from the renormalisation function of the gluon $Z_3^{(1)}$ and the non-abelian part of $Z_1^{(1)}$, which we call $Z_1^{(1)}|_{\text{n.a.}}$. The two depend on the gauge, but this gauge dependence cancels in the combination

$$\beta_0 = -2Z_g^{(1)} = Z_3^{(1)} - 2Z_1^{(1)}|_{\text{n.a.}}. \quad (162)$$

For instance, in the Feynman gauge

$$Z_3^{(1)} = \frac{\alpha_s}{\epsilon} \frac{5C_A - 2n_f}{12\pi}, \quad Z_1^{(1)} = -\frac{\alpha_s}{\epsilon} \frac{C_F + C_A}{4\pi}, \quad (163)$$

where $\alpha_s = g_s^2/(4\pi)$ and $n_f$ is the number of massless (a.k.a. “active”) quark flavours contributing to the renormalisation of the gluon propagator. This gives

$$\beta_{\text{QCD}}(\alpha_s) = -\frac{11C_A - 2n_f}{12\pi} \alpha_s^2 = -\frac{21}{12\pi} \alpha_s^2, \quad (164)$$

where the latter expression corresponds to the actual value of the beta function for $n_f = 6$ active flavours, as is the case at very high momentum scales. The fact that the beta function of QCD is negative when $\alpha_s$ is small means that the QCD coupling decreases with energy. This property is known as asymptotic freedom, and is crucial to be able to compute hadronic cross sections in terms of quarks and gluons. In fact, when probed at short distances, hadrons appear as made up of pointlike constituents, quarks and gluons, which interact very feebly. Therefore, the Feynman rules we have learnt so far are enough to compute high-energy observables, for instance jet cross sections, as will be explained in the phenomenology course. At larger distances, the QCD coupling becomes stronger and stronger, at a point that quarks and gluons bind together to form hadrons. This phenomenon is known as confinement.
Summary

This has been a very quick tour through some very important, deep and interesting ma-
terial. I hope it has provided some insight into the quantum field theory descriptions of
QED and QCD, and provided you with useful tools for the future.

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THE STANDARD MODEL

Dr Fedor Bezrukov (University of Manchester)
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1 Global and local symmetries

The Standard Model is the gauge theory with SU(3)$_c \times $ SU(2)$_L \times $ U(1)$_Y$ gauge group, with spontaneous symmetry breaking. It is known, that the only good (renormalizable) quantum field theory that gives rise to the massive non-abelian vector bosons is a gauge theory with spontaneous symmetry breaking, or Higgs mechanism. We will review the construction of the abelian and non-abelian symmetries, and the mechanism of symmetry breaking. Then, we will also describe the interactions with the matter fermions (leptons and quarks).

1.1 Global Abelian U(1)

The simplest example of global symmetry is the Abelian U(1) symmetry. For a complex scalar field it is given by

$$\phi(x) \rightarrow \phi'(x) = e^{i\alpha}\phi(x),$$

and the invariant Lagrangian density (up to the operators of dimension 4) is

$$\mathcal{L}[\phi(x), \partial_\mu \phi(x)] = \partial_\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{2}(\phi^* \phi)^2.$$  

Let us note the consequences of this symmetry. First, it tells that the masses of the real scalar fields defined as $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ are equal, $m_1 = m_2 = m$. Second, by Noether theorem, there is a conserved current

$$j_\mu = i\phi^* \partial_\mu \phi - i\phi \partial_\mu \phi^*.$$  

If the fields satisfy the equations of motion, then $\partial_\mu j_\mu = 0$, and charge $Q \equiv \int d^3 x j_0$ does not change with time.
1.2 Gauge (local) $U(1)$

If we would like to have local symmetry

$$\phi(x) \rightarrow \phi'(x) = e^{i\alpha(x)}\phi(x),$$

then the kinetic term in (2) stops being invariant. This can be compensated by promotion of the derivative to the covariant derivative

$$D_\mu \phi = (\partial_\mu - ieA_\mu)\phi.$$  \hspace{1cm} (5)

Requirement that transformation of $D_\mu \phi$ is covariant (just multiplication by the phase, as for the field $\phi$ itself)

$$D_\mu \phi \rightarrow (D_\mu \phi)' = e^{i\alpha(x)}(\partial_\mu \phi + i\phi \partial_\mu \alpha - ieA'_\mu \phi)$$

$$= e^{i\alpha(x)}(\partial_\mu \phi - ieA_\mu \phi) = e^{i\alpha(x)}D_\mu \phi,$$

defines the transformation of the gauge field $A_\mu$

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \alpha.$$  \hspace{1cm} (7)

The gauge invariant expression constructed out of the gauge field is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  \hspace{1cm} (8)

This leads to the local $U(1)$ symmetric action (scalar electrodynamics)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu \phi)^*(D_\mu \phi) - V(\phi^*\phi).$$  \hspace{1cm} (9)

Note, that we still have two equal mass scalar degrees of freedom, two polarisations for the massless vector field, and conserved charge.

The case of fermionic matter field was discussed in detail in QED lectures.

1.3 Global non-Abelian

The next generalization is to use more complex symmetries for the action. The simplest example is the system with $N$ scalar fields

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$  \hspace{1cm} (10)

and the transformation

$$\phi_i(x) \rightarrow \phi'_i(x) = \omega_{ij}\phi_j(x),$$  \hspace{1cm} (11)
with $\omega_{ij} \in \text{SU}(N)$ (we assume implicitly sum over the repeated index $j$). $\text{SU}(N)$ is the group of unitary $N \times N$ matrices $\omega \omega^\dagger = 1$ with $\det \omega = 1$. Unitary transformation leaves the scalar product $(\phi, \phi) \equiv \phi_i^* \phi_i$ invariant, so the following Lagrangian is invariant

$$\mathcal{L} = \partial_\mu \phi_i^* \partial_\mu \phi_i - m^2 \phi_i^* \phi_i - \lambda (\phi_i^* \phi_i)^2.$$  \hfill (12)

In a similar way a symmetric action can be written for a group of fermion fields.

As a consequence of the symmetry all the fields have the same mass, the same coupling constant. Also, in agreement with the Noether theorem there is a set of conserved currents (or charges).

I should mention, that generalization of this construction is possible for more complicated groups or group representations—then in general, instead of using the element $\omega \in G$ of a group $G$ ($\text{SU}(N)$ in our example) we should use a matrix $T(\omega)$ which transforms the fields according to representation $T$ of the group.

### 1.4 Gauge (local) non-Abelian

The Lagrangian \cite{12} is no longer invariant under coordinate dependent transformations

$$\phi(x) \rightarrow \phi'(x) = \omega(x) \phi(x) \hfill (13)$$

$$\omega(x) \in \text{SU}(N), \hfill (14)$$

the terms with derivatives of $\omega(x)$ emerge from the kinetic term

$$\partial_\mu \phi_i'(x) = \omega(x) \partial_\mu \phi_i(x) + \partial_\mu \omega(x) \cdot \phi(x). \hfill (15)$$

The remedy is the same as for the abelian case—introduce the gauge fields and upgrade all derivatives to covariant ones,

$$D_\mu \phi = \partial_\mu \phi - ig A_\mu \phi.$$  \hfill (16)

which transform in a covariant way under the gauge transformations

$$(D_\mu \phi)' = \omega D_\mu \phi.$$  \hfill (17)

Comparing the left and right parts of this equation gives us the required transformation rule for the gauge fields

$$D_\mu \phi' = \partial_\mu \phi' - ig A_\mu' \phi' = \omega \partial_\mu \phi + \partial_\mu \omega \phi - ig A_\mu' \omega \phi$$

$$A_\mu \rightarrow A_\mu' = \omega A_\mu \omega^{-1} + \frac{i}{g} \omega \partial_\mu \omega^{-1}.$$  \hfill (17)

The gauge field is an element of the Lie algebra of the symmetry group:

$$A_\mu = \sum_i A_\mu^i T_i,$$
where $T_i$ are generators of the group. For example, for the case of SU(2), which will be of the most interest for us, there are three generators $T^i = \tau^i/2$, with $\tau^i$ being the Pauli matrices

$$
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Thus, the Lagrangian

$$
\mathcal{L} = (D_\mu \phi)^\dagger D_\mu \phi - m^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2,
$$

is invariant under gauge transformations (13,17). The kinetic term for the gauge bosons is a generalization of the Abelian case. However, now the field strength

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + igA_\mu^A A_\nu^B c^{ijk} T_k,
$$

is no longer completely gauge invariant, but transforms in a covariant way

$$
F_{\mu\nu}(x) \rightarrow F'_{\mu\nu}(x) = \omega(x) F_{\mu\nu} \omega^{-1}(x). \quad (18)
$$

Then, the kinetic term takes the form

$$
\mathcal{L}_A = -\frac{1}{2} \text{Tr} F_{\mu\nu} F_{\mu\nu} = -\frac{1}{4} F_{\mu\nu}^i F_{\mu\nu}^i. \quad (19)
$$

The resulting theory has a set of conserved currents, and a lot (equal to the number of generators of the Lie algebra) massless vector bosons. Due to the commutator in the definition of the field strength tensor these gauge bosons interact between each other, making the dynamics quite different from the Abelian (QED) case.

## 2 Spontaneous symmetry breaking

### 2.1 Global symmetry—Goldstone theorem

For now, we are still far from the interesting goal—getting a theory of massive vector bosons. The next attempt is to take a Lagrangian with some internal symmetry, and break it spontaneously. That is, find a Lagrangian which is invariant, but which has a vacuum state which is not invariant under the symmetry transformation. The simplest example is to take the $\text{U}(1)$ symmetric theory (2) (written for better visualisation using two real fields, instead of one complex field), but select a rather peculiar potential

$$
V(\phi_1, \phi_2) = -\frac{\mu^2}{2} (\phi_1^2 + \phi_2^2) + \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2.
$$

The minimum of this potential is not at zero, but at any point of the circle $\phi_1^2 + \phi_2^2 = v^2 \equiv \mu^2/\lambda$. Note, that symmetry transformation (rotation in the plane $\phi_1, \phi_2$) takes a point on this circle and moves it to another point—so any given vacuum is not invariant...
under the symmetry. To study the theory we should first select one of the vacuums (not really important, which one), for example

\[ \phi_1 = v, \quad \phi_2 = 0. \]

Then, we can expand the fields on top of this vacuum

\[ \begin{align*}
\phi_1(x) &= v + \chi(x) \\
\phi_2(x) &= \theta(x).
\end{align*} \tag{20} \]

and study the dynamics of the small perturbations \( \chi(x), \theta(x) \). The expansion of the kinetic part is trivial while the potential is more complicated

\[ V = -\mu^2 \left[ (v + \chi)^2 + \theta^2 \right] + \frac{\lambda}{4} \left[ (v + \chi + \theta)^2 \right]^2 + \frac{\mu^4}{4\lambda}. \]

Up to the quadratic order we get

\[ \mathcal{L}^{(2)}_{\chi, \theta} = \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{1}{2} \partial_\mu \theta \partial^\mu \theta - \mu^2 \chi^2. \]

So, we have got one massive field \( \chi \), and one massless field \( \theta \). Also, we have got a number of higher order terms, leading to various interactions between \( \chi \) and \( \theta \). This massless field is called Goldstone boson. It is a generic consequence of spontaneous breaking of the global symmetry. The Goldstone theorem tells, that for each broken generator of a global symmetry one massless boson appears in the spectrum of the perturbations.

There is an approximate physical example – pions are approximate Goldstone bosons of global SU(2) symmetry in QCD with massless quarks, which is broken by non-perturbative quark condensate. They are not exactly massless, because of small quark masses present in the fundamental theory, but a much lighter than all other states in the theory.

### 2.2 Gauge symmetry—Higgs mechanism

More interesting situation is realized if an attempt is made to spontaneously break gauge symmetry. Let us see how this happens on the simplest example of Abelian Higgs model. We will take the model from the previous section (we switch again to the complex field notations which are more natural for the charged fields) with the addition of the gauge fields

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - \left[ -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 \right], \tag{21} \]

where, as before, the transformation rules are

\[ \begin{align*}
A_\mu(x) &\rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x) \\
\phi(x) &\rightarrow \phi'(x) = e^{i\alpha(x)} \phi(x).
\end{align*} \]
We can select the vacuum (minimum of the energy of the system) by setting the gauge field to zero and using the results of the previous section for the scalar field

\[ A^{(v)}_{\mu} = 0, \quad \phi^{(v)} = \frac{1}{\sqrt{2}} v. \]  

(22)

Following standard rules the fields should be expanded into small perturbations on top of the vacuum

\[ \phi(x) = \frac{1}{\sqrt{2}} (v + \chi(x) + i\theta(x)). \]  

(23)

The covariant derivative in terms of \( A_\mu, \chi, \theta \) looks like

\[ D_\mu \phi = \frac{1}{\sqrt{2}} (\partial_\mu \chi + i\partial_\mu \theta - ie v A_\mu). \]

Substituting this expression into the action (21) and expanding to the quadratic order we get

\[ L^{(2)} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (\partial_\mu \chi)^2 - \mu^2 \chi^2 + \frac{e^2 v^2}{2} \left( A_\mu - \frac{1}{ev} \partial_\mu \theta \right)^2. \]  

(24)

Note, that \( \theta \) and \( A_\mu \) enter only in the combination

\[ B_\mu = A_\mu - \frac{1}{ev} \partial_\mu \theta, \]

so we can rewrite the Lagrangian for small perturbations using only \( B_\mu \) and \( \chi \)

\[ L^{(2)} = -\frac{1}{4} B_{\mu\nu}^2 + \frac{e^2 v^2}{2} B_\mu B_\mu + \frac{1}{2} (\partial_\mu \chi)^2 - \mu^2 \chi^2, \]  

(25)

where \( B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \) is the field strength tensor. This is the theory for two massive fields with masses

\[ m_V = ev = \frac{e}{\sqrt{\lambda}} \mu \]

\[ m_\chi = \sqrt{2\lambda} \mu. \]

No trace of the field \( \theta \) left here. No massless Goldstone bosons appeared, contrary to the SSB of a global theory.

The absence of one field deserves a closer analysis. Let us count the number of propagating degrees of freedom in the theory. If the potential does not lead to the SSB (\( -\mu^2 > 0 \), see section [1.2]) we have two massive scalar fields, and one massless vector field. The latter has only two polarizations (remember the electrodynamics), overall 4 d.o.f. If the symmetry is broken (\( \mu^2 > 0 \)) we have only one massive scalar field, and a massive vector field, which has 3 degrees of freedom (we can no longer use gauge symmetry to remove the longitudinal polarization). Overall, in both cases the system has 4 d.o.f. This is the essence of the Higgs mechanism—the Goldstone boson that would have been associated with the symmetry breaking is getting “eaten” by the gauge boson. The result is exactly what we are looking for—a theory with a massive vector...
bosons. Moreover, the theory still has gauge theory as the origin, which leads to good UV properties of the theory. At low energies the remains of the gauge symmetry can be seen in the relations between interaction constants and masses of the theory.

Note, that we could use gauge invariance for a simpler way to obtain the interaction Lagrangian of the physical degrees of freedom. As far as all configurations connected by gauge transformation are completely equivalent, we could define a gauge

\[
\text{Im} \phi = 0 \\
\text{Re} \phi = \frac{1}{\sqrt{2}}(v + \chi).
\]

This gauge is called \textit{unitary} and has no trace of the unphysical degree of freedom. Writing the action in this gauge immediately gives the spectrum and interaction vertices for all the physical particles of the theory. However, analysis of the quantum radiative corrections in this gauge is rather involved, because the original gauge structure is very well hidden.

The generalization of the Higgs mechanism to non-abelian symmetries we can study directly on the real world example—Standard Model.

3 Standard Model \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y

3.1 Gauge and Higgs bosons

Let us start form the construction of the real full bosonic sector of the Standard Model. We want two gauge symmetries, weak \text{SU}(2)_L (‘L’ stands for left, which reflects its relation to the fermionic degrees of freedom), and \text{U}(1) which is called hypercharge. The corresponding fields will be \( F^a_\mu \), with \( a = 1, 2, 3 \), corresponding to three \text{SU}(2) generators, and \( B_\mu \). We will also add one complex doublet

\[
\Phi = \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right).
\]

The gauge invariant Lagrangian is

\[
\mathcal{L} = -\frac{1}{4} F^a_{\mu \nu} F^a_{\mu \nu} - \frac{1}{4} B_{\mu \nu} B_{\mu \nu} + (D_\mu \Phi)^\dagger D_\mu \Phi - \lambda \left( \Phi^\dagger \Phi - \frac{v^2}{2} \right)^2,
\]

where the field strength tensors are

\[
F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g e^{abc} A^b_\mu A^c_\nu,
\]

\[
B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,
\]

and the covariant derivative is

\[
D_\mu \Phi = \partial_\mu \Phi - i g \frac{q}{2} A^a_\mu \lambda^a - i g' \frac{q'}{2} B_\mu \Phi,
\] (26)
Here $g$ and $g'$ are coupling constants for the two gauge groups. Note, that the scalar transforms in fundamental representation of $SU(2)_L$, and is also charged with the charge $g'/2$ under the $U(1)$.

The vacuum configuration can be chosen as
\[
A^a_\mu = B_\mu = 0 \\
\Phi = \left( \begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} \end{array} \right) \equiv \Phi^{(v)}.
\] (27)

Note, that this configuration is still invariant under some specific gauge transformations. Really, the group transformations corresponding to the generator
\[
Q = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)
\]
would not change the vacuum
\[
Q\Phi^{(v)} = 0
\] (28)
(if the generator annihilates the vacuum, then the group transformation of the form $\exp(i\alpha Q)$ would leave the vacuum intact). This specific generator is is a combination of the generators of the $U(1)$ hypercharge rotation and the third $SU(2)_L$ generator
\[
Q = T^3 + Y/2,
\] (29)
where $Y$ is the hypercharge operator, which is proportional to the unit matrix for all fields, but may have different value for different fields. Therefore, we expect that the corresponding $U(1)$ group will remain unbroken, leading to the electromagnetic interaction with massless photon.

The simplest way to analyse the spectrum of the theory is to fix the unitary gauge
\[
\Phi = \left( \begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} + \frac{\chi}{\sqrt{2}} \end{array} \right).
\] (30)

Really, it can be easily verified that arbitrary (non-zero) doublet can be transformed by an $SU(2)$ transformation to this form. The covariant derivative is
\[
D_\mu \Phi = D_\mu \Phi + \left[ -\frac{ig}{2} A^1_\mu \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) - \frac{ig}{2} A^2_\mu \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right) - \frac{ig}{2} A^3_\mu \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) - \frac{ig'}{2} B_\mu \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right] \Phi.
\]

In the unitary gauge we get
\[
D_\mu \Phi = \left( \begin{array}{c} -\frac{ig}{2\sqrt{2}} (A^1_\mu - i A^2_\mu) (v + \chi) \\ -\frac{i}{2\sqrt{2}} (g' B_\mu - g A^3_\mu) (v + \chi) + \frac{1}{\sqrt{2}} \partial_\mu \chi \end{array} \right).
\] (31)
It is convenient to introduce instead of $A^{1,2}$ complex fields

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu)$$

so that $(W^-_\mu)^* = W^+_\mu$. Let us also introduce two vector fields

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gA^3_\mu - g'B_\mu)$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gB_\mu + g'A^3_\mu).$$

Fields $Z_\mu$ and $A_\mu$ are chosen in a way so that the covariant derivative (31) contains only $Z_\mu$, and the following normalization condition is satisfied

$$Z^2_\mu + (A_\mu)^2 = (A^3_\mu)^2 + B^2_\mu.$$  

Finally, the covariant derivative (31) takes the form

$$D_\mu \phi = \left(\begin{array}{c}-i\frac{g}{2}W^+_\mu \\
\frac{1}{\sqrt{2}}\partial_\mu \chi + i\frac{g^2 + g'^2}{2\sqrt{2}}vZ_\mu
\end{array}\right) + \left(\begin{array}{c}-\frac{i}{2}W^+_\mu \chi \\
\frac{1}{\sqrt{2}}Z_\mu \chi
\end{array}\right),$$

Here the first column is linear in excitations (fields $W^\pm_\mu$, $\chi$, $Z_\mu$), and the second is of the second order. The contribution of the covariant derivative to the quadratic part of the Lagrangian is then given by

$$[(D_\mu \Phi)^\dagger D_\mu \Phi]^{(2)} = \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{4}g^2 v^2 W^+\mu W^-\mu + \frac{1}{2}\left(\frac{g^2 + g'^2}{4}\right)Z^2_\mu.$$  

The quadratic parts of the kinetic terms for the gauge fields is expressed in the usual form in terms of the new fields $W, Z, A$ because the redefinitions were orthogonal. Finally, collecting all terms quadratic in the small perturbations gives

$$\mathcal{L}^{(2)} = -\frac{1}{2}W^\pm_{\mu\nu}W^-_{\mu\nu} + m^2_W W^+\mu W^-\mu - \frac{1}{4}F^-_{\mu\nu}F^-_{\mu\nu}$$

$$- \frac{1}{4}Z^-_{\mu\nu}Z^-_{\mu\nu} + \frac{m^2_Z}{2}Z^\mu Z_\mu$$

$$+ \frac{1}{2}(\partial_\mu \chi)^2 - \frac{m^2_\chi}{2}\chi^2,$$

where the field strengths $W^\pm_{\mu\nu}$, $F^-_{\mu\nu}$, and $Z^-_{\mu\nu}$ are obtained in the standard way from $W^\pm_\mu$, $A_\mu$, $Z_\mu$, and the masses are

$$m_W = \frac{gv}{2}$$

$$m_Z = \frac{\sqrt{g^2 + g'^2}v}{2}$$

$$m_\chi = \frac{\sqrt{2}\lambda v}{2}.$$
The Lagrangian describes one massive scalar field with the mass $m_\chi$, massless photon $A_\mu$, two massive (complex) vector bosons $W^\pm_\mu$ with mass $m_W$, and massive vector boson $Z_\mu$ with the mass $m_Z$.

It is convenient also to introduce the Weinberg mixing angle as

$$
\cos \theta_W = \frac{g}{\sqrt{g^2 + g'^2}},
$$
$$
\sin \theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}.
$$

Then the relations (32), (33) take the explicit form of a rotation

$$
Z_\mu = \cos \theta_W A^3_\mu - \sin \theta_W B_\mu,
$$
$$
A_\mu = \cos \theta_W B_\mu + \sin \theta_W A^3_\mu
$$

and also we get the important relation between the weak vector boson masses

$$
m_Z = \frac{m_W}{\cos \theta_W},
$$

Experimental value of $\sin \theta_W$ is $\sin^2 \theta_W = 0.23$, and it can be measured independently of the vector boson masses in the interactions of photons and $W$ and $Z$ bosons with quarks and leptons. The equation (38) is satisfied with very good precision and provides a test of the Standard Model.

Expansion of the Lagrangian beyond the quadratic level gives the interactions of the bosons, with all the vertices expressed from the three coupling constants $g, g', \lambda$ and vacuum expectation value $v$.

For completeness, we should mention the QCD gauge kinetic term

$$
\mathcal{L}_{\text{QCD}} = -\frac{1}{2} \sum_{a=1}^{8} G^a_\mu G^{\mu a},
$$

which was described in detail in the corresponding lectures. The $SU(3)$ group is not broken, the corresponding gauge bosons—gluons—are massles, but are self-interacting.

### 3.2 Leptons

In the SM there were only left handed ($V - A$) charged currents. Thus, left and right components of the spinors enter differently in the SM action. We can extract the left
and right components out of the usual four spinor using the projectors
\[ \psi_L \equiv P_L \psi \equiv \frac{1 - \gamma^5}{2} \psi, \quad \psi_R \equiv P_R \psi \equiv \frac{1 + \gamma^5}{2} \psi. \]
(39)

Note, that \( \overline{\psi}_L = (\psi_L)^\dagger \gamma^0 = \psi^\dagger \frac{1 - \gamma^5}{2} \gamma^0 = \overline{\psi} \frac{1 + \gamma^5}{2} = \overline{\psi} P_R. \) In the Weyl basis for gamma matrices \( \gamma^5 \) is diagonal, and the meaning of these definitions boils down to selecting upper two or lower two components of the 4-spinor for the left or right part. In Dirac basis (which is often used in non-relativistic limit) the meaning of (39) is less obvious.

The left electron \( e_L \) and neutrino \( \nu_L \) form a doublet under gauge transformation \( U = \exp (-i \frac{e}{2} \Delta^a(x)) \in SU(2)_L \)
\[ \chi_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad \chi_L \rightarrow \chi_L' = U \chi_L. \]

The Dirac conjugate spinors should transform in a conjugate way, that is \( \overline{\chi}_L = (\overline{\nu}_L, \overline{e}_L), \) \( \overline{\chi}_L \rightarrow \overline{\chi}_L U^\dagger = \overline{\chi}_L U^{-1}. \)

They also transform under hypercharge with charge \( Y_L = -1: \)
\[ \chi_L \rightarrow \exp(i Y_L \alpha(x)) \chi_L. \]

For the normal massive electron we need also the right handed component. It does not transform at all under \( SU(2)_L \), and has hypercharge \( Y_R = -2 \)
\[ e_R \rightarrow \exp(i Y_R \alpha(x)) e_R. \]

According to the formula for the electric charge \( Q = T^3 + Y/2 \) we get \( Q = -1 \) for both electron components, and 0 for neutrino, as needed.

We can now readily write the kinetic terms for the lepton Lagrangian.
\[ L_{\text{kin}} = \overline{\chi}_L i \gamma^\mu D_\mu \chi_L + \overline{e}_R i \gamma^\mu D_\mu e_R, \]
(40)

where the covariant derivatives are
\[ D_\mu \chi_L = (\partial_\mu - i g A^a_\mu T^a - i \frac{g'}{2} Y_L B_\mu) \chi_L, \]
(41)
\[ D_\mu e_R = (\partial_\mu - i \frac{g'}{2} Y_R B_\mu) e_R. \]
(42)

The only terms that are allowed by the \( SU(2)_L \times U(1)_Y \) symmetries and contains only fermions of the form \( \overline{\psi} \psi \) and not more than one scalar field are
\[ L_{\text{Yukawa}} = -G_e \overline{\chi}_L \Phi^a e_R - G_e \overline{e}_R \Phi^a_L \chi_L, \]
(43)

where repeated \( SU(2) \) index \( a \) implies summation. Note, that the terms are trivially \( SU(2)_L \) invariant. The sum of hypercharges for this term is zero, so it is \( U(1)_Y \) invariant. The requirement \( -Y_L + Y_R = 0 \) constraints possible hypercharges that can be
assigned to the fields. Note also, that terms of the form $\bar{\chi}_{L} \chi_{L}$ are trivially zero because of the chirality projectors.

The kinetic part (40) gives rise to the normal kinetic terms for the fermions, and to the interactions with all the gauge bosons—charge, neutral and electric currents. Writing explicitly in components (40) using (41) and (42) we get for the charged current

$$\mathcal{L}_{CC} = -\frac{g}{\sqrt{2}} (J_{\mu}^{+} W_{-}^{\mu} + J_{\mu}^{-} W_{+}^{\mu}),$$

$$J_{\mu}^{\pm} = \bar{\nu}_{L} \gamma_{\mu} \nu_{L}.$$

To get the neutral and electromagnetic current we also have to express $A_{3}$ and $B_{3}$ via $Z$-boson and photon fields using (37), which gives

$$\mathcal{L}_{EC} = eA_{\mu} \bar{e} \gamma_{\mu} e,$$

$$\mathcal{L}_{NC} = \frac{g}{2 \cos \theta_{W}} Z_{\mu} ( -\bar{\nu}_{L} \gamma_{\mu} \nu_{L} - \bar{e}_{L} \gamma_{\mu} e_{L} + 2 \sin^{2} \theta_{W} \bar{e} \gamma_{\mu} e),$$

where the electric charge is

$$e \equiv g \sin \theta_{W}.$$

The Yukawa interaction part in the Lagrangian (43) after symmetry breaking gives both mass term for electrons, and electron interaction with the Higgs boson. It is easiest to see this in the unitary gauge (30), which leads to

$$\mathcal{L}_{\text{Yukawa}} = -\frac{1}{\sqrt{2}} G_{e} (v + H) \bar{e}_{L} e_{R} - \frac{1}{\sqrt{2}} G_{e} (v + H) \bar{e}_{R} e_{L} \equiv -\frac{1}{\sqrt{2}} G_{e} (v + H) \bar{e} e,$$

leading to the electron mass $m_{e} = G_{e} v / \sqrt{2}$, and electron Higgs interaction with the strength $G_{e} / \sqrt{2} = m_{e} / v$. Note, that the interaction strength is directly proportional to the mass of the particle, so that this interaction is practically absent for the electron, but is sizeable for $\tau$-lepton.

### 3.3 Quarks

The left-handed components of the quarks also form a doublet under $SU(2)_{L}$, however, as far as the electric quark charges are different from leptons, the choice of the hypercharge is different

$$Q_{L} = \begin{pmatrix} U_{L} \\ D_{L} \end{pmatrix}, \quad Y_{Q} = 1/3.$$

The right handed components are still $SU(2)_{L}$ singlets, but as far as all the quarks are Dirac massive fermions, we need right-handed counterparts to both up and down components of the left-handed doublet, $U_{R}$ and $D_{R}$ with hypercharges $Y_{U} = 4/3$, and $Y_{D} = -2/3$. Also, all of the quark fields, $Q_{L}, U_{R},$ and $D_{R}$ have $SU(3)$ colour index $i = 1, 2, 3$, which we don’t write explicitly here (see QCD lectures for details).
The kinetic quark terms in the Lagrangian are, as usual,
\[ L_{\text{kin}} = \bar{Q}_{L} i\gamma_{\mu} D_{\mu} Q_{L} + \bar{U}_{R} i\gamma_{\mu} D_{\mu} U_{R} + \bar{D}_{R} i\gamma_{\mu} D_{\mu} D_{R}, \]  
and
\[ D_{\mu} Q_{L} = (\partial_{\mu} - ig A_{\mu}^{a} T^{a} - ig' Y_{Q} B_{\mu} - ig s G_{\mu}) Q_{L}, \]  
\[ D_{\mu} U_{R} = (ig' Y_{U} B_{\mu} - ig s G_{\mu}) U_{R}, \]  
\[ D_{\mu} D_{R} = (ig' Y_{D} B_{\mu} - ig s G_{\mu}) D_{R}. \]  
(50)

Similarly to the lepton sector, this term generates the normal quark kinetic terms, and gauge boson interactions. The SU(3) strong force interactions are described in QCD lectures, while the electroweak currents in the interaction terms (44), (45), and (46) get the following contributions
\[ J^{+}_{\mu} = \bar{U}_{L} \gamma_{\mu} D_{L}, \]  
\[ J^{EM}_{\mu} = Q_{U} \bar{U} \gamma_{\mu} U + Q_{D} \bar{D} \gamma_{\mu} D, \]  
\[ J^{NC}_{\mu} = \sum_{\text{fermions}} \bar{\psi}(\gamma_{\mu}(1 - \gamma^{5})T^{3} - 2Q \sin^{2} \theta_{W}) \psi, \]  
(54)

where \( T^{3} = 1/2, Q = 2/3 \) for the up quarks, \( T^{3} = -1/2, Q = -1/3 \) for the down quarks. Note, that this is a generic formula, that can be used for leptons also.

The Yukawa term is more complicated for quarks. The term similar to (43) gives mass to the down quark, while to generate mass for the up component the Higgs doublet should have non-zero value in the upper component, instead of the lower component. There are two ways to achieve this. One is to introduce another Higgs doublet \( \Phi^{(2)} \), which in the vacuum is equal to \( \left( \frac{v}{\sqrt{2}} \right) \). This is actually the situation that is realized in SUSY, which imposes significant constraints on the possible Lagrangians. Without SUSY, in plain SM, the mass of the up quarks can be generated without addition of new scalar fields. This is possible because the doublet \( \Phi^{c}_{a} \equiv \epsilon_{ab} \Phi^{*}_{b} \), where \( \epsilon_{ab} \) is the antisymmetric tensor, and we have explicitly written the SU(2) indexes, transforms under the fundamental representation of SU(2) (to insure the gauge invariance we had to use the complex conjugate field in this expression, which is impossible according to the SUSY rules). Then, the Yukawa Lagrangian for quarks can be written as
\[ L_{\text{Yukawa}} = -G_{D} \bar{Q}_{L} \Phi D_{R} - G_{U} \bar{Q}_{L} \Phi^{c} U_{R}, \]  
(57)

and using the unitary gauge expressions (30) and \( \Phi^{c} = \frac{1}{\sqrt{2}} \left( v + H^{(2)} \right) \) we get masses both for up and down quarks, and proportional interactions with the Higgs boson.

One more complication arises when the construction is generalized to several generations of matter. In general, all the Yukawa constants \( G_{e}, G_{D}, \) and \( G_{U} \) can become arbitrary matrices in flavour. Thus, the mass matrices \( m_{u,ij} = G_{u,ij} v/\sqrt{2}, m_{d,ij} = G_{d,ij} v/\sqrt{2}, \) that are generated after symmetry breaking are not diagonal. They can be diagonalized.

\[ \text{Mathematically this corresponds to the observation that fundamental and conjugate representations are isomorphic in case of SU(2).} \]
by separate orthogonal redefinitions of the left and right up and down quarks,

\[
\begin{align*}
    u_{L,i} &= V_{uL,ij} \tilde{u}_{L,j}, \\
    u_{R,i} &= V_{uR,ij} \tilde{u}_{R,j}, \\
    d_{L,i} &= V_{dL,ij} \tilde{d}_{L,j}, \\
    d_{R,i} &= V_{dR,ij} \tilde{d}_{R,j},
\end{align*}
\]

where \( V_{\ldots} \) are orthogonal matrices. With this transformation the masses can be made diagonal, the kinetic terms, electromagnetic and neutral currents stay diagonal, but the charged current changes

\[
J_\mu^+ = \bar{u}_L \gamma_\mu V_{fg} d_L^f ,
\]

where \( f, g = 1, 2, 3 \) are flavour indexes, and \( V_{fg} \equiv V^{-1}_{uL} V_{dL} \) is the unitary CKM (Cabibbo-Kobayashi-Maskawa) matrix. In particular, it has quite strong mixing between the first two generations for the charged current

\[
V_{fg} \simeq \begin{pmatrix}
    \cos \theta_c & \sin \theta_c & 0 \\
    -\sin \theta_c & \cos \theta_c & 0 \\
    0 & 0 & 1
\end{pmatrix},
\]

where the Cabibbo angle \( \theta_c \simeq 19^\circ \), and other off-diagonal elements are of the order \( 10^{-2} \).

A general \( 3 \times 3 \) unitary matrix is parametrized by 9 real parameters. However, constant phase rotation of quark fields makes 5 of them unobservable, so there are only 4 physical parameters in the CKM matrix—three mixing angles and one phase.

\[
V_{fg} = \begin{pmatrix}
    c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\
    -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & -c_{13} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} \\
    s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13}
\end{pmatrix},
\]

where \( c_{ij} \equiv \cos \theta_{ij}, s_{ij} = \sin \theta_{ij} \), and \( \theta_{ij}, i, j = 1, 2, 3 \) are mixing angles, and \( \delta_{13} \)—CP-violating phase.

### 3.4 Parameter count

Overall SM has 19 parameters. The electroweak scalar sector is parametrized by 4 parameters—\( v, \lambda, g, g' \) are the parameters in the Lagrangian, which map to equivalent observables with more physical sense

\[
M_Z = 91.2 \text{ GeV}, \quad M_H = 125 \text{ GeV},
\]

\[
\alpha \equiv \frac{e^2}{4\pi} = \frac{1}{137}, \quad \sin^2 \theta_W = 0.231.
\]

Note, that other parameters, like \( M_W \) can be expressed from these 4. One more parameter \( \alpha_S \equiv \frac{g_2^2}{4\pi} \) defines strong interactions.

For the fermions we need 9 masses for 3 charged leptons and 6 quarks. The quark sector has CKM mixing matrix with 3 angles and one CP phase.
There is one more parameter, QCD phase \( \theta_{QCD} \), which we left in our discussion. It appears in the action in the form of an additional gauge invariant term

\[
\mathcal{L}_{QCD \ phase} = \int d^4x \frac{\theta_{QCD}}{16\pi} \text{Tr} \, G_{\mu\nu}G^{\mu\nu}.
\]

The term is actually a full derivative, so it does not change the equations of motion. However, there are non-perturbative effects where it contributes. Via these effects this term leads to potentially observable CP-violating effects, like neutron dipole moment.

### 3.5 Full SM after SSB

Let us just summarize the full SM. The whole action is rather short before spontaneous symmetry breaking

\[
\mathcal{L} = -\frac{1}{2} \text{Tr} \, G_{\mu\nu}G^{\mu\nu} - \frac{1}{2} \text{Tr} \, W_{\mu\nu}W^{\mu\nu} - \frac{1}{2} \text{Tr} \, B_{\mu\nu}B^{\mu\nu} + \sum_{\psi=\chi_{L,R},q_{L,R},D_{L,R}} i\bar{\psi} \gamma^{\mu} D_\mu \psi + \sum_{f,g} \left[ -G_{f}^{g} \chi_{L}^{f} \Phi e_{g}^{f} - G_{D}^{f} Q_{L}^{f} \Phi D_{R}^{g} - G_{U}^{f} Q_{L}^{f} \Phi U_{R}^{g} \right] + (D_\mu \Phi)\dagger D_\mu \Phi - \lambda \left( \Phi^{\dagger} \Phi - \frac{v^2}{2} \right)^2.
\]

The Lagrangian after the SSB is much longer, and the fact that it originated from a gauge symmetric action is well hidden. However, all the particle interactions can be directly read from it. The SM Lagrangian after symmetry breaking contains the following parts:

\[
\mathcal{L} = \mathcal{L}_{QCD} + \mathcal{L}_{\text{lept}} + \mathcal{L}_{f,\text{EM}} + \mathcal{L}_{f,\text{weak}} + \mathcal{L}_{Y} + \mathcal{L}_{V} + \mathcal{L}_{H} + \mathcal{L}_{VH}.
\]

The QCD part is

\[
\mathcal{L}_{QCD} = -\frac{1}{4} G_{\mu\nu}^{a} G^{\mu\nu}_{\mu\nu} + \sum_{\text{quarks}} \bar{q}(i\gamma^{\mu} \partial_{\mu} - m_{q} - g_{s} \frac{\lambda^{a}}{2} \gamma^{\mu} G_{\mu}^{a}) q,
\]

where we sum over all quarks. The part describing the free leptons is

\[
\mathcal{L}_{\text{lept}} = \sum_{\gamma=e,\mu,\tau} \bar{\nu}_{\gamma}(i\gamma^{\mu} \partial_{\mu} - m_{\nu_{\gamma}}) \nu_{\gamma} + \sum_{g} \bar{\nu}_{g}(i\gamma^{\mu} \partial_{\mu} P_{L} \nu_{g},
\]

where the sum is over three generations, and only left neutrino components enter (in pure SM neutrino are massless). The electromagnetic part sums over all charged fermions \( f \)

\[
\mathcal{L}_{f,\text{EM}} = eA_{\mu} \sum_{f} Q_{f} \bar{f} \gamma^{\mu} f,
\]
where $Q_f$ is the electric charge of the fermion in units of $e$. The weak interactions of the leptons are described by

$$\mathcal{L}_{\text{weak}} = \frac{g}{2\sqrt{2}} W_\mu \sum_g \bar{\nu}_g \gamma^\mu (1 - \gamma^5) e_g + \text{h.c.}$$

$$+ \frac{g}{2\sqrt{2}} W_\mu \sum_{f,g} \bar{u}_f \gamma^\mu (1 - \gamma^5) V_{fg} d_g + \text{h.c.}$$

$$+ \frac{g}{2\cos \theta_W} Z_\mu \sum_{\text{fermions}} \bar{f} \gamma^\mu (T_3^f (1 - \gamma^5) - 2Q_f \sin^2 \theta_W) f,$$

where $T_3^f$ is 1/2 for neutrinos and upper quarks, and $-1/2$ for charged leptons and down quarks. The fermion Higgs interactions are proportional to masses

$$\mathcal{L}_Y = \sum_{\text{fermions}} \frac{m_f}{v} \bar{f} f \cdot H.$$

Careful expansion of the kinetic terms of the gauge fields gives

$$\mathcal{L}_V = -\frac{1}{4} (F_{\mu
u})^2 - \frac{1}{4} (Z_{\mu
u})^2 + \frac{m_Z^2}{2} Z_\mu Z_\mu$$

$$- \frac{1}{2} W^-_{\mu\nu} W^+_{\mu\nu} + m_W^2 W^-_{\mu\nu} W^+_{\mu\nu} + \frac{g^2}{4} (W^+_{\mu\nu} W^-_{\mu\nu} - W^+_\mu W^-_\mu)^2$$

$$- \frac{ig}{2} (F^\mu_\nu \sin \theta_W + Z^\mu_\nu \cos \theta_W) (W^-_{\mu\nu} W^+_\nu - W^+_\mu W^-_\nu),$$

where

$$W^-_{\mu\nu} \equiv (\partial_\mu - i e A_\mu - ig \cos \theta_W Z_\mu) W^-_\nu - (\mu \leftrightarrow \nu).$$

The Higgs is described the scalar theory with cubic quartic interactions

$$\mathcal{L}_H = \frac{1}{2} (\partial_\mu H)^2 - \frac{m_H^2}{2} H^2 - \lambda v H^3 - \frac{\lambda}{4} H^4.$$

Note, that the vev and self-coupling define both mass and cubic interaction terms. Finally, there are interactions between the Higgs and vector bosons

$$\mathcal{L}_{HV} = \frac{g^4}{4} v H W^-_\mu W^+_\mu + \frac{g^2 + g'^2}{4} v H Z_\mu Z_\mu + \frac{g^2}{4} H^2 |W^-_\mu|^2 + \frac{g^2 + g'^2}{8} H^2 Z_\mu Z_\mu.$$

### 3.6 Symmetries of the SM

The main starting point for the formulation of the SM action was the invariance under gauge symmetries. At the same time, SM has more (global) symmetries.
3.6.1 Baryon number

Simultaneous phase rotation of all quarks (and opposite rotation of antiquarks)

\[ q \rightarrow e^{i\beta/3}q, \quad \bar{q} \rightarrow e^{-i\beta/3}\bar{q} \]

leaves all the terms in the action invariant. By Noether theorem this leads to the conserved current

\[ j^B_\mu = \frac{1}{3} \sum_q \bar{q} \gamma_\mu q \]

and, respectively, the conserved baryon number

\[ B = \int d^3x j_0 = \frac{1}{3} (N_q - N_{\bar{q}}) = N_{\text{baryons}} - N_{\text{anti-baryons}}. \]

The immediate consequence is that the lightest baryon–proton–should be stable. Therefore, evidence of proton decay would show deviations form the SM.

3.6.2 Lepton numbers

In a similar way lepton numbers can be defined. Moreover, in the absence of neutrino masses there are no mixing between the lepton generations, and lepton numbers can be defined individually for each generation:

\[ L_e = N_e + N_{\nu_e} - (N_{\bar{e}} + N_{\bar{\nu}_e}), \]

\[ L_\mu = N_\mu + N_{\nu_\mu} - (N_{\bar{\mu}} + N_{\bar{\nu}_\mu}), \]

\[ L_\tau = N_\tau + N_{\nu_\tau} - (N_{\bar{\tau}} + N_{\bar{\nu}_\tau}). \]

Similarly, searches for violation of lepton number, like \( \mu \rightarrow e\gamma \) decays, would indicate deviations from SM.
COLLIDER PHENOMENOLOGY

Dr Simon Badger (IPPP, University of Durham)
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1 Introduction

Historically the lecture notes for the phenomenology course have consisted of the slides presented in the lectures. These notes are intended to provide additional information, and more mathematical detail, on the more theoretical aspects of the course which don't change from year to year. The recent experimental results, which as the LHC experiments take more and more data change from day-to-day, will continue to be presented solely on the slides used in the lectures.

These notes have been adapted from notes from Daniel Maître and Peter Richardson.

In order to study hadron collisions we need to understand the basics of cross section calculations, Quantum Chromodynamics (QCD) and jets which we will first consider in the simpler environment of $e^+e^-$ and lepton-hadron collisions before we go on to study hadron–hadron collisions.

Unfortunately there is no single good book on modern phenomenology. Two old classics but now a bit dated are:

- Quarks and Leptons Halzen and Martin [1];
- Collider Physics Barger and Phillips [2].

Two good books, although mainly focused on QCD and probably at a bit too high a level for this course, are:

- QCD and Collider Physics Ellis, Stirling and Webber [3];
- Quantum Chromodynamics Dissertori, Knowles and Schmelling [4];

and of course the classic on Higgs physics

- The Higgs Hunter’s Guide Gunion, Haber, Kane and Dawson [5].

In addition the recent reviews:

- Towards Jetography [6] which provides a good primer on jet physics;
- General-purpose event generators for LHC physics [7] which gives a detailed description of the physics of Monte Carlo event generators;

are good sources of additional information.
2 e\(^{+}\)e\(^{-}\) Annihilation

While electron-positron colliders are less relevant for current phenomenology than they were before, they are a good starting point to discuss many concepts one also finds at hadron colliders.

If we consider what happens when electrons and positrons collide, then the most likely thing is that some hadrons are produced. However, none of the Lagrangians or Feynman rules you’ve learnt involve hadrons. This is the key issue in most collider physics, we can calculate things for quarks and gluons but we observe hadrons.

2.1 Leading Order

We will start by studying one of the simplest possible processes, e\(^{+}\)e\(^{-}\) annihilation via the exchange of a photon or Z\(^{0}\) boson, as shown in Fig. 1. This process can produce either quarks or leptons. Unfortunately due to quark confinement we cannot observe free quarks directly, instead quarks and antiquarks will produce hadrons with unit probability. Much of what we will study in this course will be concerned with the question, given that we observe hadrons how do we infer what was going on in the fundamental process involving quarks?

We will start with the simplest example. Given that quarks and antiquarks produce hadrons with unit probability we can measure the cross section for the process e\(^{+}\)e\(^{-}\) \(\rightarrow\) q\(\bar{q}\), which we can calculate perturbatively, by measuring the cross section for e\(^{+}\)e\(^{-}\) \(\rightarrow\) hadrons. This is the case because gluons (which also produce hadrons) do not couple directly to the leptons. This is the basis of most collider phenomenology, we want to measure things using hadrons that we can calculate using partons. The total cross section for e\(^{+}\)e\(^{-}\) annihilation into hadrons is the simplest such observable.

Using the techniques you have learnt in the other courses you can now calculate the total cross section for e\(^{+}\)e\(^{-}\) annihilation. In reality it is more common to study the ratio

\[
R \equiv \frac{\sigma(e^{+}e^{-} \rightarrow \text{hadrons})}{\sigma(e^{+}e^{-} \rightarrow \mu^{+}\mu^{-})},
\]

as this reduces experimental uncertainties. At low energies this process is dominated by photon exchange so we can neglect the Z\(^{0}\) boson. In this limit

\[
\sigma(e^{+}e^{-} \rightarrow \mu^{+}\mu^{-}) = \frac{4\pi\alpha^{2}}{3s},
\]
where $s$ is the centre-of-mass energy of the collision squared. The cross section for the production of quarks is

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{3s} \sum_q e_q^2 N_c,$$

where $e_q$ is the charge of the quark in units of the positron charge and the sum runs over all quarks for which the centre-of-mass energy $\sqrt{s} > 2m_q$, where $m_q$ is the mass of the quark. Remember we must sum over all the quantum numbers of the quarks so the cross section is multiplied by number of colours, $N_c$. Therefore for centre-of-mass energies much less than the mass of the $Z^0$ boson, $\sqrt{s} \ll M_Z$,

$$R = \sum_q e_q^2 N_c = N_c \left( \frac{4}{9} + \frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9} \right) .$$

The expected picture is shown in figure 2. The experimental measurement of this ratio is shown in Fig. 3 as a function of energy showing the thresholds for the production of the charm and bottom quarks. Below the charm threshold there are three active quarks down ($e_d = -\frac{1}{3}$), up ($e_u = \frac{2}{3}$) and strange ($e_s = -\frac{1}{3}$) giving $R = 2$. Above the charm ($e_c = \frac{2}{3}$) threshold $R = \frac{10}{3}$ while above the bottom ($e_b = -\frac{1}{3}$) threshold $R = \frac{11}{3}$.

### 2.1.1 The Z resonance

For energies $\sqrt{s} \sim m_Z$ we will need to include the effects of the second diagram in Fig. 1. The cross-section will then have three different contributions, the photon background, the
Figure 3: The ratio \( R \equiv \frac{\sigma(e^+e^-\rightarrow\text{hadrons})}{\sigma(e^+e^-\rightarrow\mu^+\mu^-)} \) as a function of energy taken from Ref. [8].

Z-boson resonance and the photon-Z interference. The total cross-section, summed and averaged over spins can be written as (e.g. [3]):

\[
\sigma (f \bar{f} \rightarrow f' \bar{f'}) = \frac{\alpha^2 \pi}{2s} \int_{-1}^{1} d(\cos \theta) \left\{ (1 + \cos^2 \theta) \left( g_f^2 g_{f'}^2 + \frac{g_Z^2}{4g_e^2} g_f q_f q_{f'} v_f v_{f'} \chi_1 + \frac{g_Z^2}{16g_e^2} (a_f^2 + v_f^2)(a_{f'}^2 + v_{f'}^2) \chi_2 \right) + \cos \theta \left( \frac{g_Z^2}{2g_e^2} a_f a_{f'} v_f v_{f'} \chi_1 + \frac{g_Z^4}{2g_e^4} a_f a_{f'} v_f v_{f'} \chi_2 \right) \right\}
\]

where

\[
g_Z = \frac{1}{\cos \theta_w \sin \theta_w} \quad \chi_1 = \frac{s(s - m_Z^2)}{(s - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \quad \chi_2 = \frac{s^2}{(s - m_Z^2)^2 + m_Z^2 \Gamma_Z^2}
\]

The axial \((v_f = T_3^f - 2q_f \sin^2 \theta_w)\) and vector \((a_f = T_3^f)\) couplings in the Standard Model are given in Table 1. \(T_3^f\) is the 3rd component of the weak isospin as covered in the course on the Standard Model. The terms proportional to \(\chi_2\) come from the Z resonance while those proportional to \(\chi_1\) come from the photon-Z interference. \(\Gamma_Z\) is the width of the Z boson.

Later we will take a closer look at the EW sector of the Standard Model and use this measurement to find constraints on the number of neutrinos families below the Z mass threshold (see Figure 28).
### 2.2 Higher Order Corrections

When we draw Feynman diagrams we are performing a perturbative expansion in the (hopefully) small coupling constant. Unfortunately the strong coupling often isn’t very small, at the $Z^0$ mass, $\alpha_S(M_Z) = 0.118$. We therefore need to consider higher orders in the perturbative expansion. There are always two types of correction:

- real gluon emission;
- virtual gluon loops.

#### 2.2.1 Real Emission

There are two possible diagrams for gluon emission, see Fig. 4.

![Feynman diagrams for $e^+e^- \to q\bar{q}g$.](image)

Considering photon exchange for simplicity, is

$$M = e^2 e_q g_s t^{ij}_a \bar{v}(p_b) \gamma_{\mu} u(p_a) \frac{-g^{\mu\nu}}{q^2} \bar{u}(p_1) \left[ \gamma_{\sigma} \frac{p_1 + p_3}{(p_1 + p_3)^2} \gamma_{\nu} - \gamma_{\nu} \frac{p_2 + p_3}{(p_2 + p_3)^2} \gamma_{\sigma} \right] v_j(p_2) \epsilon^a_{\alpha}(p_3),$$

where $p_{a,b}$ are the 4-momenta of the incoming electron and positron, respectively. The outgoing quark, antiquark and gluon have 4-momenta $p_{1,2,3}$, respectively. The total momentum of the system $q = p_a + p_b = p_1 + p_2 + p_3$. The gluon has colour index $a = 1, \ldots, N_c^2 - 1$ whereas the quark/antiquark have colour indices $i, j = 1, \ldots, N_c$.

Summing/averaging over spins and colours

$$\overline{|M|^2} = \frac{4e^2 e_q g_s^2 N_c}{s} C_F \left( \frac{p_1 \cdot p_3}{p_1 \cdot p_3} + \frac{(p_1 \cdot p_b)^2}{p_2 \cdot p_3} + \frac{(p_2 \cdot p_a)^2}{p_1 \cdot p_3} \right).$$

The colour algebra gives a colour factor

$$\sum_a t^{a}_{ij} (t^{a}_{ij})^* = t^{a}_{ij} t^{a}_{ji} = \frac{1}{2} \delta^{aa} = \frac{1}{2} (N_c^2 - 1) = N_c C_F,$$
where the colour charges in the fundamental (quarks and antiquarks) and adjoint (gluons) representations are

\[ C_F \equiv \frac{1}{2N_c}(N_c^2 - 1) \quad \text{and} \quad C_A \equiv N_c, \]  

(8)

respectively. More about the colour algebra can be found in appendix C.

The three-body phase space is

\[ d\Phi_n(p_a + p_b; p_1, p_2, p_3) = \frac{(2\pi)^4 \delta^{(4)}(p_a + p_b - p_1 - p_2 - p_3)}{(2\pi)^3 2E_1 (2\pi)^3 2E_2 (2\pi)^3 2E_3} \]

\[ = \frac{1}{8(2\pi)^5} E_1 dE_1 d\cos \theta d\phi E_2 dE_2 d\cos \beta d\alpha \frac{1}{E_3} \delta(\sqrt{s} - E_1 - E_2 - E_3), \]

where \( \theta \) and \( \phi \) are the polar and azimuthal angles, respectively, of the outgoing quark with respect to the beam direction. The polar and azimuthal angles of the antiquark with respect to the quark direction are \( \beta \) and \( \alpha \), respectively. We have integrated over \( p_3 \) using the \( \delta \)-function and assumed that the outgoing particles are massless.

Using momentum conservation

\[ E_3 = |p_3| = |p_1 + p_2| = \sqrt{E_1^2 + E_2^2 + 2E_1E_2 \cos \beta}. \]  

(9)

Therefore the integral over the remaining \( \delta \)-function is

\[ \int d\cos \beta \delta(\sqrt{s} - E_1 - E_2 - E_3) = \frac{E_3}{E_1E_2}, \]  

(10)

so

\[ d\Phi_n(p_a + p_b; p_1, p_2, p_3) = \frac{1}{8(2\pi)^5} E_1 dE_1 d\cos \theta d\phi E_2 dE_2 d\cos \beta d\alpha \]  

\[ = \frac{s}{16(2\pi)^2} dx_1 dx_2 \frac{d\cos \theta d\phi d\alpha}{2(2\pi)^2}, \]

where \( x_i \equiv 2E_i/\sqrt{s} \). Momentum and energy conservation requires that \( x_1 + x_2 + x_3 = 2 \).

The total cross section is

\[ \sigma = \frac{1}{2s} \frac{s}{16(2\pi)^2} \int dx_1 dx_2 \frac{d\cos \theta d\phi d\alpha}{2(2\pi)^2} |M|^2, \]  

(12)

\[ = \frac{4\pi \alpha^2 e_q^2 N_c}{3s} C_F \frac{\alpha s}{2\pi} \int dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}. \]

The contribution from the \( Z^0 \) boson is the same except for \( \sigma_0 \). This is divergent at the edge of phase space as \( x_{1,2} \to 1 \) so that the total cross section is \( \sigma = \infty! \)

This is a common feature of all perturbative QCD calculations. Configurations which are indistinguishable from the leading-order result are divergent. Physically there are two regions where this happens.
1. **Collinear limit**: If we take \( x_1 \to 1 \) at fixed \( x_2 \) or \( x_2 \to 1 \) at fixed \( x_1 \). We can see what happens physically by considering the dot product of the antiquark and gluon 4-momenta, *i.e.*

\[
2p_2 \cdot p_3 = \frac{s x_2 x_3}{2} (1 - \cos \theta_{23}) = s (1 - x_1) \Rightarrow (1 - \cos \theta_{23}) = \frac{2(1 - x_1)}{x_2 x_3} \to 0. \quad (13)
\]

So the limit \( x_1 \to 1 \), where the matrix element diverges, corresponds to the angle between the antiquark and gluon \( \theta_{23} \to 0 \), *i.e.* collinear emission of the gluon from the antiquark. Similarly the limit \( x_2 \to 1 \) corresponds to collinear emission of the gluon from the quark.

2. **Soft limit**: \( x_{1,2} \to 1 \) at fixed \( \frac{1 - x_1}{1 - x_2} \). We can consider what happens in this limit by considering the energy of the gluon

\[
E_g = \frac{\sqrt{s}}{2} x_3 = \frac{\sqrt{s}}{2} (1 - x_1 + 1 - x_2) \to 0,
\]

*i.e.* the matrix element diverges in the soft limit, when the energy of the gluon is small.

These are both universal features of QCD matrix elements. In general one can see how the divergencies appear by looking at the propagator just before the emission of a gluon.

\[
P^2 = (k + p)^2 = 2|\vec{k}||\vec{p}|(1 - \cos \theta)
\]

From this expression one can see that the propagator vanishes (and therefore divergences appear) when the gluon is either soft \((|k| \to 0)\) or collinear \((\cos \theta \to 0)\).

In these limits QCD matrix elements *factorize, i.e.* the matrix element including the emission of a soft or collinear gluon can be written as the convolution of the matrix element before the emission and a universal term describing collinear or soft emission.

**Collinear Limit**  If we first consider collinear emission we take the momentum of the gluon \( p_3 \) parallel to \( p_2 \) (\( \theta_{23} = 0 \)). We can therefore define

\[
p_2 = (1 - z)\bar{p}_2, \quad p_3 = z\bar{p}_2, \quad \text{with} \quad \bar{p}_2^2 = 0, \quad (15)
\]

where \( \bar{p}_2 \) is the momentum of the antiquark before the gluon radiation and \( z \) is the fraction of the original antiquark’s momentum carried by the gluon. In this limit the matrix element factorizes

\[
|M_{q\bar{q}g}|^2 = |M_{q\bar{q}}|^2 \times \frac{g_s^2}{p_2 \cdot p_3} \times C_F \frac{1 + (1 - z)^2}{z}. \quad (16)
\]
As does the phase space
\[ dx_1 dx_2 \rightarrow \frac{1}{4} z(1-z) dz d\theta_{23}^2. \] (17)

Putting this together
\[
\sigma = \sigma_0 \int \frac{d\theta_{23}^2}{2\pi} dz C_F \frac{\alpha_S}{\pi} \frac{1 + (1-z)^2}{z} = \sigma_0 \int \frac{d\theta_{23}^2}{2\pi} dz \frac{\alpha_S}{\pi} \hat{P}_{q \to gg}(z). \] (18)

The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) splitting function is a universal probability distribution for the radiation of a collinear gluon in any processes producing a quark. The splitting functions are:

\[
\hat{P}_{g \to gg}(z) = C_A \left[ \frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right]; \quad \hat{P}_{q \to gg}(z) = C_F \frac{1+z^2}{1-z}; \] (19)

\[
p_{q \to gg}(z) = T_R \left[ z^2 + (1-z)^2 \right]; \quad \hat{P}_{q \to gg}(z) = C_F \frac{1+(1-z)^2}{z}; \]

where \( z \) is the fraction of the momenta carried by the first outgoing particle and \( T_R = \frac{1}{2} \).

**Soft Limit** In the limit that \( E_g \to 0 \) the matrix element for the process factorizes

\[
M_{q \bar{q}g} = M_{q \bar{q}g} s^a t_{ij} \left( \frac{p_1}{p_1 \cdot p_3} - \frac{p_2}{p_2 \cdot p_3} \right) \cdot \epsilon_A(p_3), \] (20)

the *eikonal current*. The matrix element squared therefore factorizes in this case

\[
|M_{q \bar{q}g}|^2 = |M_{q \bar{q}}|^2 g_s^2 C_F \frac{2p_1 \cdot p_2}{p_1 \cdot p_3 p_2 \cdot p_3}. \] (21)

The phase space is

\[
dx_1 dx_2 \rightarrow \frac{2}{s} E_g dE_g d\cos \theta. \] (22)

So in the soft limit

\[
\sigma = \sigma_0 \int C_F \frac{\alpha_S}{2\pi} \frac{dE_g}{E_g} d\cos \theta \frac{2(1 - \cos \theta_{qg})}{(1 - \cos \theta_{qg})(1 - \cos \theta_{qg})}. \] (23)

the *dipole radiation pattern* a universal probability distribution for the emission of a soft gluon from any colour-connected pair of partons.\(^1\)

Figure 5 illustrates the dipole radiation pattern.

### 2.2.2 Virtual Corrections

There are three diagrams involving virtual gluon loops, see Fig. 6. This contribution is also divergent, but negative. This will cancel the real divergence to give a finite answer. To show this we need to regularize both the real and virtual cross sections and add them together. The result should be finite when we remove the regularization. The standard

\(^1\)Strictly this is only universal at the amplitude level, not as a probability distribution.
way of doing this is to work in $d = 4 - 2\epsilon$ dimensions where to regularize these infrared divergences $\epsilon < 0$. In this case

$$
\sigma_{\text{real}} = \sigma_0 C_F \frac{\alpha_S}{2\pi} H(\epsilon) \left( \frac{4}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon) \right),
$$

$$
\sigma_{\text{virtual}} = \sigma_0 C_F \frac{\alpha_S}{2\pi} H(\epsilon) \left( -\frac{4}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon) \right),
$$

where $H(0) = 1$. The sum

$$
\sigma_{\text{total}} = \sigma_{\text{real}} + \sigma_{\text{virtual}} = \sigma_0 C_F \frac{3\alpha_S}{4\pi},
$$

is finite as $\epsilon \to 0$. So finally combining this correction with the leading-order result

$$
R(e^+e^-) = R_0(e^+e^-) \left( 1 + \frac{\alpha_s}{\pi} \right).
$$

Measuring $R(e^+e^-)$ is one way of measuring the strong coupling giving

$$
\alpha_S(m_Z) = 0.1226 \pm 0.0038.
$$

The second and third order corrections, and the results for the next-to-leading-order corrections including quark masses are also known.

This is the simplest example of an observable which we can calculate using perturbation theory involving quarks and gluons, but measure experimentally using hadrons. We now need to go on and consider more complicated observables.

\footnote{Taken from the Ref. [8].}
3 Running Coupling

In addition to the infrared, soft and collinear, divergences we saw in the calculation of \( \sigma(e^+e^- \rightarrow \text{hadrons}) \) it is possible to have ultraviolet divergences. The virtual corrections shown in Fig. 7 are divergent in the ultraviolet. These, and other similar corrections, lead to the strong coupling being renormalized to absorb the ultraviolet singularities. The renormalisation procedure introduces an unphysical renormalisation scale \( \mu \).

The leads to:

1. diagrams are dependent on \( \mu \);
2. \( \alpha_S \) is replaced by the running coupling \( \alpha_S(\mu) \);
3. although we can’t calculate the coupling we can calculate how it changes with scale:

\[
\mu^2 \frac{d\alpha_S}{d\mu^2} \equiv \beta(\alpha_S) = -\beta_0 \alpha_S^2 + \ldots \quad \beta_0 = \frac{11N_c - 4T_R n_f}{12\pi},
\]

where \( n_f \) is the number of active quark flavours.

For \( \beta_0 > 0 \) the coupling displays asymptotic freedom, i.e. \( \alpha_S(\mu) \rightarrow 0 \) as \( \mu \rightarrow \infty \) which allows us to perform perturbative calculations at high energies where the coupling is small.

It is standard to quote the value of \( \alpha_S(M_Z) \). The value at other scales can by found by solving the evolution equation. Recent experimental measurements of the strong coupling evolved to the \( Z^0 \) mass and the running of coupling are shown in Fig. 8.

It is common to define a scale \( \Lambda_{QCD} \) so that

\[
\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln \left( \frac{\mu^2}{\Lambda_{QCD}^2} \right)} [1 + \ldots].
\]

In general there is a choice of precisely how we perform the renormalisation, which leads to both renormalisation scale and scheme dependence. Physical observables don’t depend on \( \mu_F \) or the renormalisation scheme, but fixed order perturbative calculations do.

3.1 Higher order calculations

Since the strong coupling constant is not very small the perturbative series converges slower than it does in QED. To get reliable QCD predictions we need at least NLO precision and NNLO is preferable for important processes, but NNLO calculations are very
challenging. Perturbative calculations for hadron colliders have two unphysical parameters: the factorisation and renormalisation scales. The former defines the separation between the perturbative and non-perturbative description of the proton and the latter is needed to remove the ultra-violet divergences and specifies at which scale the coupling constant should be evaluated. This dependence is an artefact of the truncation of the perturbative series, if we were able to compute the entire perturbative series to all orders, the dependence would drop out. Therefore the dependence on the factorisation and renormalisation scales is used as a gauge of the theoretical error due to the missing orders.

3.2 Infrared safety

To enable a meaningful comparison between theory and experiment it is important that the observable is defined in a way that allows the perturbative prediction to be carried out at higher orders. One requirement is that the observable should be infrared safe. By this we mean that the value of the observable does not change in the case of a collinear splitting or in the case of the emission of a soft particle. Mathematically it means that the observable $\mathcal{O}$ has to fulfill the following properties. For a collinear splitting of the parton with momentum $p_i$ we need

$$\mathcal{O}(p_1, ..., p_i, ..., p_n) = \mathcal{O}(p_1, ..., z p_i, (1 - z) p_i, ..., p_n)$$

and in the case of a parton’s momentum $p_j$ becoming soft we require

$$\mathcal{O}(p_1, ..., p_i, p_j, p_k, ..., p_n) \rightarrow \mathcal{O}(p_1, ..., p_i, p_k, ..., p_n)$$

for $p_j \rightarrow 0$. 

---

Figure 8: Measurements of the strong coupling at the $Z^0$ mass and the running of the coupling taken from Ref. [8].
Examples of infrared unsafe observables or procedures are

- number of partons
- observables using incoming parton momentum fractions
- observables based on older jet algorithms
- using infrared unsafe observables as renormalisation or factorisation scale

It is not always easy to find out whether an observable/procedure is infrared safe, in order to so so correctly we will need to study the details of the jet clustering algorithm and the factorisation of the initial state in hadron collisions. The factorisation of short and long distance effects for hadronic initial states is covered in Section 4 while Section 6 covers details of different jet algorithms.

### 3.3 Event Shapes

If we consider the $e^+e^-$ annihilation events shown in Fig. 9 we see a collimated bunch of hadrons travelling in roughly the same direction as the original quarks or gluons. Often you can “see” the jets without some fancy mathematical definition. We will come back and consider jets in more detail when we consider hadron–hadron collisions later in the course, in Section 6.

An alternative to defining jets is to define a more global measure of the event which is sensitive to the structure of the event. We need a number of properties to achieve this, the most important of which is infrared safety, i.e. if there is soft or collinear emission the answer doesn’t change. Formally if a parton splits into two collinear partons

$$ p \rightarrow zp + (1-z)p, \quad (29) $$
Figure 10: Phase space for $e^+e^- \to q\bar{q}g$. The requirement that $x_3 \leq 1$ ensures that $x_1 + x_2 \geq 1$ by momentum conservation so that physical phase space is the upper half plane.

or if a soft parton is emitted with momentum

$$p \to 0,$$  \hspace{1cm} (30)

the result should not change.

After the total cross section, the simplest infrared safe observable is the thrust

$$T = \max \sum_i \frac{|\vec{p}_i \cdot \hat{n}|}{\sum_i |\vec{p}_i|},$$  \hspace{1cm} (31)

where the sum is over all the final-state particles and the direction of the unit vector $\hat{n}$, the thrust axis, is chosen to maximize the projection of the momenta of the final-state particles along that direction.

For a two-jet pencil-like event all the particles lie along the thrust axis giving $T = 1$. For a totally spherical event the thrust can be calculated by taking a spherical distribution of particles in the limit of an infinite number of particles giving $T = \frac{1}{2}$. For three partons the thrust axis will lie along the direction of the most energetic parton, by momentum conservation there is an equal contribution to the thrust from the other partons giving $T = \max \{x_1, x_2, x_3\}$.

In order to calculate the differential cross section with respect to the thrust for $e^+e^- \to q\bar{q}g$ we can start from the differential cross section in Eqn.12. In many cases when we wish to introduce a new quantity into a differential cross section it is easier to insert the definition using a $\delta$-function rather than performing a Jacobian transform, in this case we use

$$1 = \int dT \delta(T - \max \{x_1, x_2, x_3\}),$$  \hspace{1cm} (32)

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to give
\[
\frac{d\sigma}{dT} = \sigma_0 C_F \frac{\alpha_s}{2\pi} \int dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \delta(T - \max\{x_1, x_2, x_3\}),
\]  
where \(\sigma_0\) is the leading-order cross section for \(e^+e^- \rightarrow q\bar{q}\). This expression can be evaluated in each of the three phase-space regions shown in Fig. 10. First in the region where \(x_1 > x_2, 3\)

\[
\left. \frac{d\sigma}{dT} \right|_{x_1 > x_2, 3} = \sigma_0 C_F \frac{\alpha_s}{2\pi} \int_{2(1-T)}^T dx_2 \frac{T^2 + x_2^2}{(1 - T)(1 - x_2)}
\]

\[
= \sigma_0 C_F \frac{\alpha_s}{2\pi} \left[ \int_{2(1-T)}^T dx_2 \frac{T^2 + 1}{(1 - x_2)} - (1 + x_2) \right],
\]

where we have used the \(\delta\)-function to integrate over \(x_1\) and the limits on \(x_2\) are given by \(x_2 = x_1 = T\) for the upper limit and \(T = x_1 = x_3 = 2 - x_1 - x_2 = 2 - T - x_2\) for the lower limit. Performing the integral gives

\[
\left. \frac{d\sigma}{dT} \right|_{x_1 > x_2, 3} = \sigma_0 C_F \frac{\alpha_s}{2\pi} \left[ (T^2 + 1) \ln \left( \frac{2T - 1}{1 - T} \right) + 4 - 7T + \frac{3}{2}T^2 \right].
\]

The same result is obtained in the region \(x_2 > x_1, 3\) due to the symmetry of the formulae under \(x_1 \leftrightarrow x_2\).

In the final region we can take the integrals to be over \(x_2, 3\) and use the \(\delta\)-function to eliminate the integral over \(x_3\) giving

\[
\left. \frac{d\sigma}{dT} \right|_{x_3 > x_1, 2} = \sigma_0 C_F \frac{\alpha_s}{2\pi} \int_{2(1-T)}^T dx_2 \frac{(2 - T - x_2)^2 + x_2^2}{(T + x_2 - 1)(1 - x_2)},
\]

\[
= \sigma_0 C_F \frac{\alpha_s}{2\pi} \int_{2(1-T)}^T dx_2 \frac{1}{T} \left[ (2 - T - x_2)^2 + x_2^2 \right] \left[ \frac{1}{T + x_2 - 1} + \frac{1}{1 - x_2} \right],
\]

\[
= \sigma_0 C_F \frac{\alpha_s}{2\pi} \frac{2}{T} \left[ (2 - 2T + T^2) \ln \left( \frac{2T - 1}{1 - T} \right) + 2T - 3T^2 \right],
\]

where after the integral over \(x_3\), \(x_1 = 2 - x_2 - T\) and the limits are calculated in the same way as before.

Putting the results from the three regions together gives

\[
\frac{d\sigma}{dT} = \sigma_0 C_F \frac{\alpha_s}{2\pi} \left[ \frac{2}{T(1 - T)} (3T(T - 1) + 2) \ln \left( \frac{2T - 1}{1 - T} \right) + \frac{3(3T - 2)(T - 2)}{1 - T} \right].
\]

This result clearly diverges as \(T \rightarrow 1\), indeed in this limit

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dT} \mid_{T \rightarrow 1} \sim C_F \frac{\alpha_s}{2\pi} \frac{4}{(1 - T)} \ln (1 - T) + \frac{3}{1 - T}.
\]

We can use this result to define a two- and three-jet rate so that the three jet rate is

\[
R_3(\tau) = \int_{\frac{1}{2}}^{1-\tau} \frac{1}{\sigma_0} \frac{d\sigma}{dT} \mid_{\tau \rightarrow 0} = C_F \frac{\alpha_s}{2\pi} 2 \ln^2 \tau,
\]
Figure 11: Thrust distribution at various centre-of-mass energies compared with Monte Carlo simulations, taken from Ref. [9].

and the two jet rate

$$R_2(\tau) = 1 - R_3(\tau) \tau^{-\alpha_s} 1 - C_F \frac{\alpha_S}{2\pi} 2 \ln^2 \tau.$$  \hspace{1cm} (40)

Similar logarithmically enhanced terms appear at all orders in the perturbative expansion giving an extra $\ln^2 \tau$ at every order in $\alpha_S$, i.e.

$$R_2(\tau) \equiv \int_{1-\tau}^1 dT \frac{1}{\sigma} \frac{d\sigma}{dT} \tau^{-\alpha_s} 1 - C_F \frac{\alpha_S}{2\pi} 2 \ln^2 \tau + \left( C_F \frac{\alpha_S}{2\pi} \right)^2 2 \ln^4 \tau + \ldots$$  \hspace{1cm} (41)

Although $\alpha_S$ is small, $\ln^2 \tau$ in large so the perturbative expansion breaks down. The solution is to resum the large $\alpha_S^n \ln^{2n} \tau$ terms to all orders giving the Sudakov Form Factor

$$R_2(\tau) \tau^{-\alpha_s} \exp \left[ -C_F \frac{\alpha_S}{2\pi} 2 \ln^2 \tau \right].$$  \hspace{1cm} (42)

This is finite (zero) at $\tau = 0$, i.e. the probability for no gluon radiation is zero. In general the Sudakov form factor gives the probability of no radiation

$$P(\text{no emission}) = \exp \left[ -\hat{P}_{\text{naive}}(\text{emission}) \right].$$  \hspace{1cm} (43)

An example of the experimental measurement of the thrust distribution is shown in Fig. 11 compared to various Monte Carlo simulations which include resummation of these large logarithmic contributions.
4 Deep Inelastic Scattering

Historically measurements of deep inelastic scattering were very important for establishing the nature of QCD. Nowadays they are mainly important for the measurement of the parton distribution functions we need to calculate all cross sections for processes with incoming hadrons. As the proton isn’t fundamental at sufficiently high energies the scattering is from the constituent quarks and gluons.

\[
\theta_{k}^\mu = (k - k')^\mu
\]

Figure 12: Deep inelastic scattering kinematics.

In deep inelastic scattering processes it is conventional to use the kinematic variables shown in Fig. 12. The struck parton carries a fraction \( x \) of the four-momentum of the incoming hadron. The four-momentum of the exchanged boson is \( q \) and the virtuality of the boson \( Q^2 = -q^2 \). Using momentum conservation

\[
 xp + q = p',
\]

where \( p' \) is the 4-momentum of the scattered quark. Therefore \((xp + q)^2 = 0 \) giving \( x = \frac{Q^2}{2pq} \). Similarly the mass of the hadronic system is \( W^2 = (p + q)^2 \). By definition \((k + p)^2 = 2k \cdot p = s \) and therefore \( y = \frac{xP}{xp} = \frac{Q^2}{x} \).

Deep inelastic scattering has \( Q^2 \gg M^2 \) (deep) and \( W^2 \gg M^2 \) (inelastic), where \( M \) is the proton mass. Historically the observation and understanding of DIS was one of the key pieces of evidence for quarks. On general grounds the cross section has the form

\[
\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{xQ^4} \left[ y^2xF_1(x, Q^2) + (1 - y)F_2(x, Q^2) \right],
\]

which parameterizes the cross section in terms of two unknown structure functions, \( F_{1,2}(x, Q^2) \). If we consider that the proton is a bound state of partons we can calculate these structure functions.

Suppose that the probability of a given type of quark carrying a fraction \( \eta \) of the proton’s momentum is \( f_q(\eta) \) the cross section for hadron scattering can be written in terms of those for partonic scattering

\[
\frac{d^2\sigma(e + \text{proton})}{dx dQ^2} = \sum_q \int_0^1 d\eta f_q(\eta) \frac{d^2\sigma(e + q(\eta p))}{dx dQ^2}.
\]
Figure 13: The reduced cross section, which is equivalent to $F_2$ up to some small corrections, measured by the H1 and ZEUS experiments from Ref. [10].

Taking the outgoing parton to be on-shell:

$$(q + \eta p)^2 = 2\eta p \cdot q - Q^2 = 0 \quad \Rightarrow \quad \eta = x.$$ 

Therefore

$$\frac{d^2\sigma(e + \text{proton})}{dx dq^2} = \sum_q f_q(x) \frac{d^2\sigma(e + q(xp))}{dq^2}. \quad (47)$$

The differential cross section for $e^+(k) + q(p) \rightarrow e^+(k') + q(p')$ via photon exchange which
dominates at low $Q^2$ for neutral current scattering is
\[
\frac{d^2\sigma(e + q(xp))}{dQ^2} = \frac{2\pi\alpha^2e_q^2}{Q^4} \left[ 1 + (1 - y)^2 \right],
\]
where $e_q$ is the charge of the quark.
So in the naive parton model
\[
2xF_1(x) = F_2(x),
\]
\[
F_2(x) = x \sum_q e_q^2 f_q(x),
\]
are functions of $x$ only, Bjorken scaling. Bjorken scaling works reasonably well, see Fig. 13, but the quantum corrections, lead to scaling violations.

![Real and virtual corrections to DIS.](image)

If we consider the $O(\alpha_S)$ corrections we have the following divergent contributions:

1. soft gluon, $E_g \to 0$;
2. gluon collinear to the final-state quark;
3. gluon collinear to the initial-state quark;
4. the virtual matrix element has a negative divergence;

The contributions from (1), (2) and (4) are indistinguishable from the tree-level configuration and the divergences cancel between the real and virtual corrections. However (3) has momentum fraction $\eta > x$ and (4) $\eta = x$ so the initial-state divergences don’t cancel.

Just as with final-state radiation in the collinear limit it can be shown that
\[
d\sigma_{q\to qg} \to d\sigma_{q\to q} \times \frac{\alpha_S}{2\pi} C_F \frac{1 + z^2}{1 - z} \frac{dt}{t} \frac{dz}{z}.
\]

Here we have the unregularized DGLAP splitting function $\hat{P}_{q\to qg}$, it is singular as $z \to 1$. The virtual contribution contains a compensating singularity at exactly $z = 1$. The
The + prescription is defined by convolution with a well defined function, \( g(z) \), such that

\[
\int_0^1 \mathrm{d}z [f(z)]_+ g(z) = \int_0^1 \mathrm{d}zf(z) \left[ (g(z) - g(1)) \right].
\]
of the production of the $Z^0$ boson, as a simple example of a hadron–hadron process, in the next section we will go on and study the physics of jets.

The calculation of the cross section for the production of an $s$-channel resonance in hadron–hadron collisions is described in more detail in Appendix A.3.1 where the cross section is given in Eqn. 128. The only dependence of the cross section on the rapidity of the $Z^0$ boson is via the PDFs, i.e. the rapidity distribution of $Z^0$ contains information on the PDFs of the partons $a$ and $b$. The higher the mass of the produced system the more central it is, see Fig. 15. The $Z^0$ boson is centrally produced in both $p\bar{p}$ and $pp$ collisions. The experimental results, for example those from the Tevatron shown in Fig. 16, are in good agreement with the theoretical predictions.

At leading order the transverse momentum of the gauge boson is zero. As before we have include real and virtual corrections, as shown in Fig. 17. In the same way as DIS the initial-state singularities must be factorized into the PDFs. At low transverse momentum we need to resum the multiple soft emissions whereas, as with the $e^+e^-$ event shapes, at large $p_T$ the fixed-order approach is more reliable. The transverse momentum of the $Z^0$ boson at the Tevatron is shown in Fig. 18.

In hadron-hadron collisions we would like at least next-to-leading order (NLO) calculations. This is the first order at which we have a reliable calculation of the cross section. If possible we would like next-to-next-to-leading order (NNLO) calculations but
that is much harder and takes a long time, e.g. $e^+e^- \rightarrow 3 \text{jets}$ was calculated at: LO in 1974 [15]; NLO in 1980 [16]; NNLO in 2007 [17]. Calculating NNLO corrections is still extremely challenging in hadron collisions, only the Drell-Yan process and $gg \rightarrow H$ are known. However, we need higher order calculations because while the factorization scale uncertainty is significantly less at NLO when compared to leading order it can still be significant, see for example the scale uncertainty on the rapidity of the $Z^0$ boson shown in Fig. 19.

6 Jets

While we can often see the jets in an event when we look at an event display we need a precise definition to perform quantitative analyzes.\(^4\) Jets are normally related to the underlying perturbative dynamics, i.e. quarks and gluons. The purpose of a jet algorithm is to reduce the complexity of the final state, combining a large number of final-state particles to a few jets, i.e.

$$\left\{ p_i \right\}^{\text{jet algorithm}} \rightarrow \left\{ j_l \right\}. \quad (55)$$

We need a number of properties to achieve this (Snowmass accord):

\(^4\)This section is based on the excellent review Towards Jetography [6].
Figure 18: Transverse momentum of the $Z^0$ boson measured by the D0 experiment at the Tevatron, taken from Ref. [13].

Figure 19: Rapidity distribution of the $Z^0$ boson for the LHC at $\sqrt{s} = 14$ TeV, taken from Ref. [14].

- simple to implement in experimental analyzes and theoretical calculations;
- defined at any order in perturbation theory and gives finite cross sections at any order in perturbation theory (i.e. infrared safe);
- insensitive to hadronization effects.

The most important of these properties is infrared safety, as with the event shapes we
considered earlier. Provided the jet algorithm is infrared safe there are a range of different approaches.

The two main types of jet algorithm are:

1. cone algorithms;
2. sequential recombination algorithms.

There is a long history to this subject with: theorists and $e^+e^-$ experimentalists generally preferring recombination algorithms for their better theoretical properties; hadron collider experimentalists preferring cone algorithms for their more intuitive picture and because applying many experimental corrections was easier. However, with the start of the LHC we have converged on a specific recombination algorithm.

6.1 Cone Algorithms

The simplest, oldest, and most intuitively appealing idea is a cone algorithm. The most widely used algorithms are iterative cone algorithms where the initial direction of the cone is determined by a seed particle, $i$. The sum of the momentum of all the particles with a cone of radius $R$, the jet radius, in the azimuthal angle $\phi$ and rapidity $y$ is then used as a new seed direction and the procedure iterated until the direction of the resulting cone is stable. In this approach the momenta of all the particles $j$ such that

$$\Delta R_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2 < R^2, \quad (56)$$

are summed. As these algorithms are almost exclusively used in hadron–hadron collisions it is normal to use the kinematically variables defined in Appendix A.1.

While this may seem simple there are a lot of complications in the details of the algorithm in particular: what should be used as the seeds; what happens when the cones obtained from two different seeds share particles, overlap. The details of the treatment of these issues can lead to problems with infrared safety, which can often be very subtle.

Consider a simple approach where we take all the particles to be seeds. If we have two partons separated in $(y, \phi)$ by twice the cone radius then two jets, with the direction

![a) Seed particles](image1.png)
![b) Jet Cones](image2.png)

Figure 20: Example of cone jets.

given by that of the original partons, are formed as shown in Fig. 20. However if there is an additional soft gluon emission between the two jets, as shown in Fig. 21, depending on
the approach we can get only one jet, i.e. the algorithm is unsafe. A simple solution was to use the midpoint between all the seeds as a seed, the midpoint algorithm. This solves the problem at this level but similar problems appear for higher multiplicities. The final solution, for the only known infrared safe cone algorithm, SISCon, is to avoid the use of seeds and treat overlapping jets carefully.

6.2 Sequential Recombination Algorithms

In this approach jets are constructed by sequential recombination. We define a distance measure between two objects $d_{ij}$, in hadron collisions we must also define a distance measure $d_{iB}$ with respect to the beam direction. There are two variants of the algorithm the inclusive where all jets are retained and exclusive where only jets above the cut-off value of the jet measure $d_{\text{cut}}$, the jet resolution scale, are kept. The algorithm proceeds as follows:

1. the distance measure is computed for each pair of particles, and with the beam direction in hadronic collisions, and the minimum found;

2. if the minimum value is for a final-state merging in the exclusive approach the particles $i$ and $j$ are recombined into a pseudoparticle if $d_{ij} \leq d_{\text{cut}}$, while in the inclusive algorithm they are always recombined;

3. otherwise if a beam merging is selected in the inclusive approach the particle is declared to be a jet, while in the exclusive approach it is discarded if $d_{iB} \leq d_{\text{cut}}$;

4. in the inclusive approach we continue until no particles remain, while in the exclusive approach we stop when the selected merging has $\min\{d_{iB}, d_{ij}\} \geq d_{\text{cut}}$.

In the inclusive approach the jets are all those selected from merging with the beam, whereas in the exclusive approach the jets are all the remaining particles when the iteration is terminated.

The choice of the distance measure, and to a lesser extent the recombination procedure,\(^5\) defines the algorithm.

\(^5\)Or sometimes pseudorapidity $\eta$.

\(^6\)In practice the so-called “E-scheme” where the four-momenta of the particles are added to give the pseudoparticle’s four-momentum is almost always used.
The earliest JADE algorithm for $e^+e^-$ collisions uses the distance measure

$$d_{ij} = 2E_iE_j (1 - \cos \theta_{ij}) ,$$

where $E_{i,j}$ are the energies of the particles and $\theta_{ij}$ the angle between them. In $e^+e^-$ collisions we have to use the exclusive algorithm and it is conventional to use a dimensionless measure $y_{ij} = d_{ij}/Q^2$, where $Q$ is the total energy in the event. While this choice can easily be proved to be safe in the soft and collinear limits there are problems with the calculation of higher order corrections.

Therefore a class of $k_T$ algorithms was developed in which the distance measure was chosen to be the relative transverse momentum of the two particles in the collinear limit, i.e.

$$d_{ij} = \min\{E_i^2, E_j^2\} \theta_{ij} \simeq k_{\perp ij}^2 \text{ for } \theta_{ij} \to 0 \text{.}$$

(58)

In $e^+e^-$ collisions the conventional choice is

$$d_{ij} = 2 \min\{E_i^2, E_j^2\} (1 - \cos \theta_{ij}) .$$

(59)

In hadron collisions it is best to use a choice which is invariant under longitudinal boosts along the beam direction. The standard choice is

$$d_{ij} = \min\{p_{i\perp}^2, p_{j\perp}^2\} \frac{\Delta R_{ij}^2}{R^2} ,$$

(60)

where $R$ is the “cone-size” and $p_{i\perp}$ is the transverse momentum of particle $i$ with respect to the beam direction. The standard choice for the beam distance is $d_{iB} = p_{i\perp}^2$. There are other definitions, particularly of the distance $d_{ij}$, which are invariant under longitudinal boosts but that in Eqn. 60 is the most common choice.

In general there is a whole class of measures defined by

$$d_{ij} = \min\{p_{i\perp}^{2p}, p_{j\perp}^{2p}\} \frac{\Delta R_{ij}^2}{R^2} ,$$

(61)

and $d_{iB} = p_{i\perp}^{2p}$.

The parameter $p = 1$ for the $k_T$ algorithm and $0$ for the Cambridge/Aachen algorithm.

Recently a new approach, the anti-$k_T$ algorithm, with $p = -1$, was proposed which favours clustering with hard collinear particles rather than clusterings of soft particles, as in the $k_T$ and Cambridge/Aachen algorithms. The anti-$k_T$ algorithm is still infrared safe and gives “conical” jets due to the angular part of the distance measure and is the algorithm preferred by both general-purpose LHC experiments.

### 6.3 Jet Cross Sections

All cone jet algorithms, expect from SISCon, are not infrared safe. The best ones typically fail in processes where we consider extra radiation from three-parton configurations while some already fail when we consider radiation from two-parton configurations, see the summary in Table 2.
Examples of the jets, and their areas, formed using different algorithms on a sample parton-level event are shown in Fig. 22. As can be seen the $k_T$ and Cambridge/Aachen algorithms tend to cluster many soft particles giving jets with an irregular area whereas the jets produced by the cone and anti-$k_T$ algorithms are more regular making applying corrections for pile-up and underlying event contamination easier.

In order to study jet production in hadron collisions we need to understand both the jet algorithm and the production of the partons which give rise to the jets. The spin/colour summed/average matrix elements are given in Table 3. Many of these matrix elements have $t$-channel dominance, typically $t \to 0 \iff p_{\perp}^2 \to 0$. As a consequence the parton–parton scattering cross section grows quickly as $p_{\perp} \to 0$ an effect which is further enhanced by the running of $\alpha_s$ when using $\mu_R = p_{\perp}$ as the renormalisation scale. An example of the $p_{\perp}$ spectrum of jets for different rapidities measured using the midpoint cone-algorithm is shown in Fig. 23.

<table>
<thead>
<tr>
<th>Process</th>
<th>Last meaningful order</th>
<th>Known at</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>JetClu Atlas cone</td>
<td>MidPoint CMS cone</td>
</tr>
<tr>
<td>inclusive jet cross section</td>
<td>LO</td>
<td>NLO</td>
</tr>
<tr>
<td>$W^\pm/Z^0 + 1$-jet cross section</td>
<td>LO</td>
<td>NLO</td>
</tr>
<tr>
<td>3-jet cross section</td>
<td>none</td>
<td>LO</td>
</tr>
<tr>
<td>$W^\pm/Z^0 + 2$-jet cross section</td>
<td>none</td>
<td>LO</td>
</tr>
<tr>
<td>jet masses in 3-jet and</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>$W^\pm/Z^0 + 2$-jet events</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Comparisons of various cone algorithms for hadron–hadron processes. Adapted from Ref. [6].

Table 3: Spin and colour summed/averaged matrix elements for $2 \to 2$ parton scattering processes with massless partons taken from Ref. [3]. A common factor of $g^4 = (4\pi\alpha_s)^2$ (QCD), $g^2e^2e_q^2$ (photon production) has been removed.
Figure 22: Examples of jets formed by different jet algorithms, taken from Ref. [6].

Figure 23: Transverse momentum spectrum of jets measured by the CDF experiment at the Tevatron, taken from Ref. [18].
6.4 Jet Properties

In general the computation of jet properties in hadron–hadron collisions is extremely complicated, however for some quantities we can get estimates of various effects. The simplest of these is to estimate the change in the $p_\perp$ between a parton and the jet it forms.

We can start by considering the change due to perturbative QCD radiation. Suppose we have a quark with transverse momentum $p_\perp$ which radiates a gluon such that the quark carries a fraction $z$ of its original momentum and the gluon a fraction $1 - z$, as shown in Fig. 24. In this case after the radiation the centre of the jet will be the parton $p_\perp (1 - z)p_\perp$

\[
\begin{align*}
\delta p_\perp = (1 - z)p_\perp - p_\perp & = -zp_\perp \quad & 1 - z > z \quad (62) \\
\delta p_\perp = zp_\perp - p_\perp & = -(1 - z)p_\perp \quad & z > 1 - z
\end{align*}
\]

with the highest transverse momentum after the branching, i.e. the quark if $z > 1 - z$ or the gluon if $z < 1 - z$. If the other parton is at an angular distance greater $\theta > R$ it will no longer be in the jet and the jet will have a smaller transverse momentum than the original parton.

We can use the splitting probabilities given in Eqn. 18 to compute the average transverse momentum loss

\[
\langle p_\perp \rangle_q = -\frac{C_F \alpha_S}{2\pi} p_\perp \int_{R^2}^1 \frac{d\theta^2}{\theta^2} \int_0^1 dz \frac{1 + z^2}{1 - z} \min\{1 - z, z\},
\]

\[
= -\frac{C_F \alpha_S}{2\pi} p_\perp \ln \left( \frac{1}{R} \right) \left[ \int_0^{\frac{1}{2}} \frac{1 + z^2}{1 - z} z + \int_{\frac{1}{2}}^1 \frac{1 + z^2}{1 - z} 1 - z \right],
\]

\[
= -\frac{C_F \alpha_S}{\pi} p_\perp \ln \left( \frac{1}{R} \right) \left[ 2 \ln 2 - \frac{3}{8} \right].
\]

The loss of transverse momentum can be calculated for gluon jets in the same way using the gluon splitting functions giving

\[
\langle p_\perp \rangle_g = -\frac{\alpha_S}{\pi} p_\perp \ln \left( \frac{1}{R} \right) \left[ C_A \left( 2 \ln 2 - \frac{43}{96} \right) + T_R n_f \frac{7}{48} \right].
\]

These calculations give

\[
\frac{\langle p_\perp \rangle_q}{p_\perp} = -0.43\alpha_S \ln \frac{1}{R}, \quad \frac{\langle p_\perp \rangle_g}{p_\perp} = -1.02\alpha_S \ln \frac{1}{R}.
\]
Figure 25: Example of various contributions to the shift of the transverse momentum, taken from Ref. [6].

So for a jet with $R = 0.4$ quark and gluon jets will have 5% and 11% less transverse momentum than the parent parton, respectively. These results are subject to significant finite $R$ and higher order corrections. The result will also depend on the precise details of the recombination scheme, for example SISCONE has a different recombination scheme where the centre of the cone is the direction of the sum of the partons and we require one parton to fall outside the cone.

While this gives the perturbative energy loss by the jet there are other effects which can change the transverse momentum of the jet. In particular the jet can also lose energy in the hadronization process and can gain energy from the underlying event.

While these effects cannot be calculated from first principles we can use some simple models to gauge the size of the effects.

One model for the effect of hadronization on event shapes in $e^+e^-$ collisions, due to Dokshitzer and Webber, is to perform a perturbative calculation and instead of stopping the calculation at some small energy scale $\mu_f$ because the strong coupling becomes non-perturbative continue the calculation into the infrared regime with a model of the strong coupling in this regime which does not diverge. They define

$$A(\mu_f) = \frac{1}{\pi} \int_{0}^{\mu_f} dk_\perp \alpha_s(k_\perp).$$  \hspace{1cm} (65)

This model can also be used to assess the size of the hadronization corrections for the jet transverse momentum. The hadronization is modelled by soft gluons with $k_\perp \sim \Lambda_{QCD}$. In this case the transverse momentum loss is

$$\delta p_\perp = z p_\perp - p_\perp = -(1 - z)p_\perp.$$  \hspace{1cm} (66)
As before the transverse momentum loss is
\[ \langle p_{\perp} \rangle_q = -\frac{C_F}{2\pi} p_{\perp} \int \frac{d\theta^2}{\theta^2} \int dz \alpha_S \frac{1 + z^2}{1 - z} (1 - z). \] (67)

As we are dealing with soft gluons \( z \sim 1 \) so \( 1 + z^2 \approx 2 \). In this case we will not use a fixed value of \( \alpha_S \) but need to evaluate it at the scale of the transverse momentum of the gluon with respect to the quark \( k_{\perp} = p_{\perp} (1 - z) \theta \). We also transform the integration variables to use \( k_{\perp} \) and \( \theta \) giving
\[ \langle p_{\perp} \rangle_q = \frac{2C_F}{\pi} \int_R^1 \frac{d\theta}{\theta^2} \int \mu k_{\perp} \alpha_S (k_{\perp}) = -\frac{2C_F A}{R}. \] (68)

Using the coefficients from fits to the \( e^+e^- \) thrust distribution
\[ \langle \delta p_{\perp} \rangle_q \sim -\frac{0.5 \text{ GeV}}{R}, \quad \langle \delta p_{\perp} \rangle_g \sim -\frac{1 \text{ GeV}}{R}. \] (69)

The hadronization correction has a \( \frac{1}{R} \) dependence on the size of the jet, unlike the \( \ln \frac{1}{R} \) dependence of the perturbative radiation.

We can estimate the underlying event contribution by assuming there is \( \Lambda_{\text{UE}} \) energy per unit rapidity due to soft particles from the underlying event giving a correction to the transverse momentum of
\[ \langle \delta p_{\perp} \rangle = \Lambda_{\text{UE}} \int_{\eta^2 + \phi^2 < R^2} d\eta \frac{d\phi}{2\pi} = \Lambda_{\text{UE}} \frac{R^2}{2}. \] (70)

This is a useful estimate although strictly the area of the jet is only \( \pi R^2 \) for the anti-\( k_T \) algorithm.

An example of the various contributions to the shift between the partonic and jet transverse momentum is shown in Fig. 25.

7 Electroweak Physics

The Standard Model has 18 parameters (assuming massless neutrinos):

- 6 quark and 3 charged lepton masses;
- 3 quark mixing angles and 1 phase;
- 1 strong coupling;
- 1 electromagnetic coupling and 3 boson masses, \( m_W, m_Z, m_h \).

All observables are a function of these 18 parameters. In principle we could choose 18 well-measured observables and define them to be the fundamental parameters of the theory, e.g.
\[ \alpha, \quad G_F, \quad \alpha_S, \quad M_Z, \quad M_h, \quad m_f. \]
and calculate everything else in terms of them.

For the electroweak part of the theory we need \( m_t, m_h \) and three other parameters to specify everything, neglecting the masses of the other Standard Model fermions. Everything else can then be calculated from these parameters, \( e.g. \)

\[
\cos \theta_W = \frac{m_W}{m_Z}, \quad e = g \sin \theta_W.
\]

The current values of the electroweak parameters are

\[
m_W = 80.41 \text{ GeV}, \quad m_Z = 91.188 \text{ GeV}, \quad \sin^2 \theta_W = 0.231, \quad G_F = 1.16639 \times 10^{-5} \text{ GeV}^{-2}.
\]

It is common to include the Fermi constant, \( G_F = \sqrt{\frac{e^2}{m_W^2}} \), from the effective theory of weak interactions at low energies as a parameter.

Different choices for the input parameters give different values for the calculated parameters.

1. input: \( \alpha(m_Z), G_F, \sin^2 \theta_W \), extracted:

\[
g = \frac{4\pi\alpha(m_Z)}{\sin^2 \theta_W} = 0.6497,
\]

\[
m_W = \frac{g}{\sqrt{4\sqrt{2}G_F}} = 79.98 \text{ GeV}, \quad m_Z = \frac{m_W}{\cos \theta_W} = 91.20 \text{ GeV};
\]

2. input: \( m_W, G_F, \sin^2 \theta_W \) extracted:

\[
m_Z = \frac{m_W}{\cos \theta_W} = 91.695 \text{ GeV},
\]

\[
g = \sqrt{4\sqrt{2}G_F m_W} = 0.653, \quad \alpha(m_Z) = \frac{g^2 \sin^2 \theta_W}{4\pi} = 1/127.51;
\]

3. input: \( m_Z, \alpha(m_Z), \sin^2 \theta_W \) extracted:

\[
m_W = \frac{m_Z}{\cos \theta_W} = 79.97 \text{ GeV}, \quad g = \frac{4\pi\alpha(m_Z)}{\sin \theta_W} = 0.6497;
\]

4. input: \( m_Z, m_W, G_F \) extracted:

\[
\sin^2 \theta_W = 1 - \left( \frac{m_W}{m_Z} \right)^2 = 0.2224,
\]

\[
g = \sqrt{4\sqrt{2}G_F m_W} = 0.653, \quad \alpha(m_Z) = \frac{g^2 \sin^2 \theta_W}{4\pi} = 1/132.42.
\]
This is due to the quantum corrections.

It was the great triumph of the LEP/SLD and Tevatron physics programmes that the quantum corrections to the theory were probed. The normal choice of input parameters is:

1. $\alpha = 1/137.035999679(94)$ the fine-structure constant at $q^2 = 0$ is accurately measured, however the error on its evolution to $q^2 = m_Z^2$ has greater uncertainty due to hadronic corrections;

2. $G_F = 1.166367(5) \times 10^5 \text{GeV}^{-2}$ is very accurately measured in muon decay $\mu^- \rightarrow e^- \nu_\mu \bar{\nu}_e$;

3. $m_Z = 91.1876 \pm 0.0021 \text{GeV}$ from the LEP1 lineshape scan;

as these are the most accurately measured.

### 7.1 Quantum Corrections to Masses

![Diagram of quantum corrections to the gauge boson propagator.](image)

Figure 26: Example quantum corrections to the gauge boson propagator.

We have already considered the running of the coupling and corrections to cross sections and other observables. However masses are also renormalized in the Standard Model. If we consider the propagator for a massive gauge boson we get corrections of the form shown in Fig. 26. If we omit the Lorentz structures this gives a propagator

\[
D(q^2) = \frac{i}{q^2 - m^2} + \frac{i}{q^2 - m^2} \frac{i\Pi(q^2)}{q^2 - m^2} \frac{i}{q^2 - m^2} + \frac{i}{q^2 - m^2} \frac{i\Pi(q^2)}{q^2 - m^2} \frac{i}{q^2 - m^2} + \ldots,
\]

where $\Pi(q^2)$ is the gauge boson self energy. This is a geometric progression, summing the series gives

\[
D(q^2) = \frac{i}{q^2 - m^2} \frac{1}{1 - \frac{i\Pi(q^2)}{q^2 - m^2}} = \frac{i}{q^2 - m^2 - \Pi(q^2)}. \tag{71}
\]

If the particle can decay to the particles in the loop there is an imaginary part of the self energy $\Pi(q^2)$ which is related to the width of the particle

\[
\text{Im } \Pi(q^2) = -iq\Gamma(q). \tag{72}
\]

The real part of the self energy correction renormalizes the particle’s mass giving

\[
D(q^2) = \frac{i}{q^2 - m_Z^2(q) + i\Gamma(q)}. \tag{73}
\]
Figure 27: Quantum corrections to the $W^\pm$ boson mass.

As we have defined to the mass of the $Z^0$ boson to be a fundamental parameter $\delta m^2_Z = 0$, by definition.

The dominant corrections to the $W$ mass come from top-bottom and Higgs loop corrections, as shown in Fig. 27.

The correction to the $W^\pm$ boson mass is

$$\delta m^2_W \sim \frac{4 s^2_W G_F}{1 - 2 s^2_W 8\pi^2/2} m_W^2 \times \frac{c^2_W N_e (m_t^2 - m_b^2)}{s^2_W} - \frac{4 s^2_W G_F}{1 - 2 s^2_W 8\pi^2/2} m_W^2 \times \frac{m_W^2 11}{3} \left( \frac{\ln \left( m^2_W \right)}{m^2_W} - \frac{5}{6} \right).$$

7.2 Electroweak Observables

A number of observables are used in the electroweak fit performed by the LEP Electroweak Working Group (LEPEWWG):

1. the $Z^0$ mass and width $m_Z, \Gamma_Z$;
2. the hadronic cross section at the $Z^0$ pole $\sigma_{\text{had}} \equiv \frac{12 \pi \Gamma(e^+ e^-) \Gamma(\text{had})}{m_Z^2 \Gamma_Z^2}$;
3. the ratio of the hadronic to leptonic partial widths of the $Z^0$, $R_\ell \equiv \frac{\Gamma(\text{had})}{\Gamma(e^+ e^-) + \Gamma(\text{had})}$, and the ratio of the bottom, $R_b \equiv \frac{\Gamma(b\bar{b})}{\Gamma(\text{had})}$, and charm, $R_c \equiv \frac{\Gamma(c\bar{c})}{\Gamma(\text{had})}$, quark partial widths to the hadronic partial width of the $Z^0$;
4. the forward-backward asymmetry for $e^+ e^- \to \bar{f}f$

$$A_{f\bar{f}}^{0, f} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B},$$

for charged leptons, $A_{e\bar{f}}^{0, e}$, bottom $A_{b\bar{f}}^{0, b}$, and charm $A_{c\bar{f}}^{0, c}$ quarks;
5. the couplings of the fermions to the $Z^0$ can be extracted from the forward-backward asymmetry in polarized scattering at SLD

$$A_{f\bar{f}}^{FB}(f) = \frac{\sigma_{LF}^f - \sigma_{LB}^f - \sigma_{RF}^f + \sigma_{RB}^f}{\sigma_{LF}^f + \sigma_{LB}^f + \sigma_{RF}^f + \sigma_{RB}^f} = \frac{3}{4} A_{f\bar{f}}.$$ \hspace{1cm} (75)

The couplings for the bottom, $A_b$, and charm, $A_c$, quarks can be extracted from these measurements. There are a number of possible ways of extracting $A_f$;
6. $\sin^2 \theta_{\text{eff}}^{\text{lept}}(Q_{f\bar{f}})$ is extracted from the hadronic charge asymmetry;
7. the $W$ mass, $m_W$, and width, $\Gamma_W$ are measured in a range of ways;

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8. The top quark mass, $m_t$, is measured at the Tevatron.

The results of the precision electroweak fit are in good agreement with the experimental results, as shown in Fig. 28, and for example shows that there are 3 massless neutrinos which couple to the $Z$ boson.

### 7.2.1 $W$ mass measurements

One of the most important quantities in electroweak sector in the mass of the $W^\pm$ boson. The first measurements of the $W$ mass were in hadronic collisions. The QCD backgrounds and resolution means that the hadronic $W^\pm$ decay mode cannot be used. The mass cannot be directly reconstructed using the leptonic mode due to the unobserved neutrino. Instead the transverse mass

$$M_{\ell\nu}^2 = 2p^\ell_{\perp}E_{\perp}(1 - \cos \phi_{\ell,\text{miss}}),$$

(76)

where $p^\ell_{\perp}$ is the transverse momentum of the observed lepton, $E_{\perp}$ is the missing transverse energy and $\phi_{\ell,\text{miss}}$ is the azimuthal angle between the lepton and the direction of the missing transverse energy, is used.

The maximum value of the transverse mass is $M_{\ell\nu}^2 \leq m_W^2$ and can be used to extract the $W^\pm$ mass. This approach was used by the UA1 and UA2 experiments for the original $W$ mass measurements and the recent results at the Tevatron, for example Fig. 29. The endpoint is smeared by the non-zero $p^\ell_{\perp}$ and width of the $W$ boson.

A major result of the LEP2 programme was the study of the production of pairs of electroweak gauge bosons, $W^+W^-$ and $Z^0Z^0$. The mass of the $W$ can be extracted in two ways:
Figure 29: The transverse mass of the $W$ at the Tevatron taken from Ref. [19].

1. measuring the cross section near the threshold

$$\sigma \sim \frac{G_F^2 m_W^2}{2\pi} \sqrt{1 - \frac{4m_W^2}{s}}.$$  

(77)

which is clean theoretical but limited by statistics, see Fig. 30;

2. reconstructing the mass from the $W$ decay products above threshold.

### 7.2.2 $\rho$ parameter

In principle we should compare the full predictions of the Standard Model, or any model of new physics, with all the electroweak observables. However it is often useful, particularly in new physics models as corrections from new particles can lead to large corrections, to consider the $\rho$ parameter. Naively

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = 1,$$

(78)

connects the $Z^0$ and $W^\pm$ masses with the weak mixing angle. The dominant loop corrections to it from self energies give

$$\Delta \rho = \frac{3G_F m_W^2}{8\sqrt{2}\pi^2} \left[ \frac{m_t^2}{m_W^2} - \frac{\sin^2 \theta_W}{\cos^2 \theta_W} \left( \ln \frac{m_H^2}{m_W^2} - \frac{5}{6} \right) + \ldots \right].$$

This relates $m_W$, $m_t$, and $m_H$. For a long time, $m_t$ was most significant uncertainty in this relation; by now, $m_W$ has more than caught up.
8 Higgs Boson

The details of the Higgs mechanism with the SM are covered in the Standard model course. In this section we summarise the properties of the Standard Model Higgs Boson that are important for hadron collider measurements.

The SM contains spin-1 gauge bosons and spin-$\frac{1}{2}$ fermions. Massless fields ensure gauge invariance under $SU(2)_L \times U(1)_Y$ and renormalizability. While we could introduce mass terms “by hand”, i.e.

\[ \mathcal{L} \propto m_A A^\mu A_\mu + m_f (\bar{\Psi}_R \Psi_L + \bar{\Psi}_L \Psi_R), \]

this violates gauge invariance. Under the gauge transformation, $A^\mu \rightarrow A^\mu + \frac{1}{g} \partial^\mu \theta$, the mass term $A^\mu A_\mu$ gives terms proportional to the gauge transformation parameter $\theta$, i.e. the gauge boson mass term is not gauge invariant. As the fields $\Psi_L$ and $\Psi_R$ transform differently under $SU(2)_L$ under the gauge transformation of the left-handed fermion field the fermion mass term is not gauge invariant.

Adding these mass terms by hand is obviously a bad idea. Instead we add a complex scalar doublet under the $SU(2)_L$ gauge group which introduces an additional four degrees of freedom. This scalar field can be coupled gauge invariantly to the gauge bosons, i.e.

\[ \mathcal{L}_{\Phi A} = (D^\mu \Phi)(D_\mu \Phi). \]
A gauge-invariant interaction term with fermions can also be included

\[ \mathcal{L}_{\Phi \Psi} = g_f \bar{\Psi}_L \Phi \Psi_R + \text{h.c.} \]  

(81)

In addition we need the Higgs potential

\[ V(\Phi) = \lambda (\Phi^\dagger \Phi)^2 + \mu^2 \Phi^\dagger \Phi. \]  

(82)

For \( \mu^2 < 0 \) this potential has an infinite number of equivalent minima,

\[ |\Phi| = \sqrt{-\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}}, \]  

(83)

as shown in Fig. 31. We expand around one of these minima giving one radial and three circular modes. The circular modes are “gauged away” → “eaten” by gauge bosons to give them mass via the vacuum expectation value (vev) the minimum of the potential.

From the structure above:

\[ (D_\mu \Phi)^2 \rightarrow g^2 v^2 W_\mu W^\mu \rightarrow M_W^2 = \frac{g^2 v^2}{2}; \]
\[ g_f \bar{\Psi}_L \Phi \Psi_R \rightarrow g_f \frac{v}{\sqrt{2}} \bar{\Psi}_L \Phi \Psi_R \rightarrow m_f = \frac{g_f v}{\sqrt{2}}; \]
\[ \lambda (|\Phi|^2 - \frac{v^2}{2})^2 \rightarrow \lambda v^2 H^2 \rightarrow M_H^2 = 2\lambda v^2. \]

This gives a fixed relation between the mass of the particles and their coupling to (surviving) scalar Higgs boson.

### 8.1 Unitarity

While in the Standard Model introducing the Higgs boson is the only way to give mass to the particles in a gauge invariant manner there are other arguments for the existence of the Higgs boson and it is interesting to ask what would happen if the Higgs boson did not exist.

---

7While we can use \( \Phi \) to couple to the down-type fermions we need to use \( i\sigma_2 \Phi^* \) to couple to the up-type fermions in a gauge invariant manner.
If we consider $W^+W^- \to W^+W^-$ scattering, via the Feynman diagrams shown in Fig. 32, in the high energy limit the matrix element is

$$M = g^2 \frac{s}{8M_W^2} \left(1 - \frac{4M_W^2}{s}\right) (1 + \cos \theta).$$

(84)

So without the Higgs boson the cross section

$$\sigma \sim \frac{s}{M_W^4},$$

(85)

for $s \gg M_W$.

This violates unitarity, so we need something to cancel the bad high energy behaviour of the cross section. We can arbitrarily invert a particle to cure this. This particle must be a scalar, suppose it has coupling, $\lambda$, to $W^+W^-$. This gives a contribution, via the Feynman diagrams in Fig. 33,

$$M = \lambda^2 \left[ -\frac{s}{8M_W^4} (1 + \cos \theta) - \frac{M_H^2}{4M_W^4} \left\{ \frac{s}{s - M_H^2} + \frac{t}{t - M_H^2} \right\} \right].$$

(86)

This cancels the bad high energy behaviour if $\lambda = gM_W$, i.e. the Higgs coupling to the $W^\pm$ boson. If we repeat the same procedure for $WW \to ZZ$ we need a coupling $g_{ZZH} \propto gm_Z$ and for $WW \to f\bar{f}$ we need a coupling $g_{f\bar{f}H} \propto gm_f$, i.e. the Higgs boson couplings to the $Z^0$ boson and Standard Model fermions.

So even if there was no Higgs boson we are forced to introduce a scalar interaction that couples to all particles proportional to their mass.

8.2 Higgs Measurements

To study the properties of the recently discovered Higgs boson we should focus our attention on,

- channels with a high signal rate;
- and a low background rate.
Table 4: Partial widths for various Higgs decay modes.

Unfortunately the channels with the highest signal rate often have the largest backgrounds. We need to be able to trigger on a given signal. Good mass resolution for the mass of the Higgs boson and its decay products can help to suppress backgrounds. We should also try and measure things that are well understood theoretically.

In order to consider the signals we need to understand how the Higgs boson is produced and then decays in hadron–hadron collisions.

The analytic results for the partial widths for various Higgs boson decay modes are given in Table 4 and the branching ratios are plotted as a function of the mass of the Higgs boson in Fig. 34. For \( m_H < 2m_W \) the Higgs boson is quite narrow, \( \Gamma_H = O(\text{MeV}) \), while for \( m_H > 2m_W \) the Higgs boson becomes obese, \( \Gamma_H(m_H = 1\text{TeV}) \approx 0.5 \text{ TeV} \). At large \( m_H \) the decay into vector boson pairs, \( W^+W^- \) and \( Z^0Z^0 \), is dominant with \( \Gamma_{H \rightarrow WW} : \Gamma_{H \rightarrow ZZ} \approx 2 : 1 \), while for small \( m_H \) the decay into bottom quark pairs is dominant.

As the Higgs boson likes to couple to heavy objects (top, \( W, Z \)) there are a range of important Higgs production processes where the Higgs boson couples to heavy particles. The Feynman diagrams for the important processes are shown in Fig. 35 while the cross sections for the important processes are shown in Fig. 36 as a function of the Higgs boson mass.

The relative importantance of different channels depend on the collider energy and the initial state (e.g. pp or \( \bar{p}p \)). At the Tevatron typical channels used for searches were:

- \( gg \rightarrow H \rightarrow W^+W^- \rightarrow \ell\ell' + E_T \) this was the “golden plated” channel because although there is no mass peak the background can be reduced by using quantities, such as the angle between the leptons, which differ in the signal and background due to the different \( W \) boson production mechanisms;

- \( q\bar{q} \rightarrow ZH \rightarrow \ell\ell b\bar{b} \) the key ingredient for this process is the \( b \)-tagging efficiency and mass resolution for jets in order to suppress the QCD backgrounds;

- \( q\bar{q}' \rightarrow WH \rightarrow \ell\nu b\bar{b} \) has similar features to \( q\bar{q} \rightarrow ZH \rightarrow \ell\ell b\bar{b} \);

- \( q\bar{q}' \rightarrow ZH \rightarrow E_T + b\bar{b} \) the key feature is again the \( b \)-tagging efficiency and mass resolution for jets in order to suppress the QCD backgrounds;

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
Decay mode & Partial Width, \( \Gamma \) \\
\hline
\( H \rightarrow f\bar{f} \) & \( \frac{G_F m_H}{8\pi\sqrt{2}} \cdot 2m_H^2 N_c \left( 1 - \frac{4m^2}{m_H^2} \right)^{\frac{1}{2}} \) \\
\hline
\( H \rightarrow W^+W^- \) & \( \frac{G_F m_H}{8\pi\sqrt{2}} \cdot m_H^2 \left( 1 - \frac{4m^2}{m_H^2} + \frac{12m^4}{m_H^2} \right) \left( 1 - \frac{4m^2}{m_H^2} \right)^{\frac{1}{2}} \) \\
\hline
\( H \rightarrow ZZ \) & \( \frac{G_F m_H}{8\pi\sqrt{2}} \cdot m_H^2 \left( \frac{4}{3m_Z^2} \right)^2 \left( \frac{4}{3} N_c Q_t^2 \right)^2 \left( 2m_t > m_H \right) \) \\
\hline
\( H \rightarrow \gamma\gamma \) & \( \frac{G_F m_H}{8\pi\sqrt{2}} \cdot m_H^2 \left( \frac{4}{3m_Z^2} \right)^2 \left( \frac{3}{4} \right)^2 \left( 2m_t > m_H \right) \) \\
\hline
\( H \rightarrow gg \) & \( \frac{G_F m_H}{8\pi\sqrt{2}} \cdot m_H^2 \left( \frac{4}{3m_Z^2} \right)^2 \left( \frac{3}{4} \right)^2 \left( 2m_t > m_H \right) \) \\
\hline
\( H \rightarrow VV^* \) & more complicated, but important for \( m_H \lesssim 2m_V \) \\
\hline
\end{tabular}
\caption{Partial widths for various Higgs decay modes.}
\end{table}
Figure 34: Branching ratios for the Higgs boson as a function of the Higgs boson mass, taken from Ref. [20], calculation by M. Spira.

- $q\bar{q}' \rightarrow W^\pm H \rightarrow W^+W^-W$; in this case there is the possibility of same sign lepton production which has a low background together with the decay of remaining $W$ to hadrons in order to increase the cross section.

Typical channels at the LHC include:

- $gg \rightarrow H \rightarrow ZZ \rightarrow 4\mu, 2e2\mu$ which is the “Golden plated” channel for $m_H > 140$ GeV, the key ingredient is the excellent resolution of the $Z$ mass peak from the leptonic decay;

- $gg \rightarrow H \rightarrow W^+W^- \rightarrow \ell\ell' + E_T$ is similar to the Tevatron analysis but with better statistics due to the larger production cross section;

- $gg \rightarrow H \rightarrow \gamma\gamma$ since Nature determined that the Higgs boson should have a mass around 120 GeV this is the easiest way to detect a Higgs boson in a collider experiment. Although the branching ratio is small, the key ingredient is the mass resolution for photon pairs and a veto on photons from $\pi^0$ decays;

- VBF$\rightarrow H \rightarrow \tau\tau$ an important mode for determining couplings to the EW sector of the SM. The key ingredient is that QCD backgrounds are reduced by requiring a rapidity gap between the two tagging jets;

- VBF$\rightarrow H \rightarrow WW$ as for VBF$\rightarrow H \rightarrow \tau\tau$;

- VBF$\rightarrow H \rightarrow b\bar{b}$ is in principle similar to the other VBF modes but it is hard to trigger on pure QCD-like objects (jets).
8.3 The effective Higgs coupling to gluons

The loop induced coupling of gluons to the Higgs boson via a massive quark loop is one of the most important ingredients for Higgs studies at a proton-proton collider. Quantum corrections to this process can be extremely large but computations with the full dependence on the quark mass are extremely difficult.

For low transverse momenta, and inclusive quantities like the total cross section, it is popular to compute the quantum corrections in an effective field theory in the limit that the top quark mass is infinitely heavy, \( m_t \to \infty \). The Lagrangian of the heavy quark Higgs effective theory (HEFT) proceeds through a dimension-five operator with the Higgs coupling directly to gluons and is derived via expansion in \( m_H/m_t \),

\[
\mathcal{L}_{\text{HEFT}} = c_{\text{HEFT}} \frac{1}{4} H G^{\mu \nu} G_{\mu \nu}
\]  

(87)

where \( c_{\text{HEFT}} = \frac{\alpha_s}{3\pi v} + \mathcal{O}(\alpha_s^2) \) for a Higgs vacuum expectation value \( v \). It is possible to
extract the value of $c_{HEFT}$ by considering to large mass limit of the triangle diagram in Figure 37. While the ratio of $m_H/m_t$ is not particularly small the approximation often does a good job at matching data when the complete results for leading order are used as a normalisation. As the transverse momentum of the produced Higgs boson increases the approximation will break down.

Computations within this effective theory are now available for the total cross section $gg \to H$ through to $N^3$LO [21, 22]. Figure 38 shows the convergence of the total cross-section by studying the dependence on the renormalisation and factorisation scale $\mu = \mu_R = \mu_F$. Further details on the approximation and other topics can be found in references [5, 23].

8.4 Extended Higgs Sectors

While current measurements show no significant deviations from a minimal SM Higgs sector adding a single Higgs doublet is the simplest choice for the Higgs sector. Many theoretically attractive models like SUSY naturally have a larger Higgs sector. However, we need to be careful to respect constraints from flavour changing neutral currents (FCNC) and the electroweak precision data.

8.4.1 The Two Higgs Doublet Model

The simplest extension to the Standard Model is the Two Higgs Doublet Model (THDM). In this model there are two Higgs doublets. There are a number of variants on the model depending on whether or not CP is conserved and how the Higgs bosons couple.
to the fermions. The most interesting variant (called Type-II) is that which occurs (in a constrained variant) in SUSY models. In the general version of the Type-II model there are $\sim 10$ new parameters, whereas in the constrained SUSY version there are only two $m_{A^0}$ and $\tan \beta$. There are indirect constraints from rare processes, e.g. kaon and bottom mixing and decays, precision EW data and cosmology.

As there are two doublets there are two vevs: $v_{1,2}$. They are constrained by the requirement

$$v_1^2 + v_2^2 = v^2 \approx (246\text{GeV})^2,$$

(88)
in order to give the correct gauge boson masses as in the Standard Model. There is an additional parameter $\tan \beta = v_2/v_1$. In the Type-II mode the $H_1$ doublet gives mass to up-type fermions while the $H_2$ doublet gives mass to down-type fermions. Both doublets couple and give mass to the gauge bosons. After electroweak symmetry breaking there are five scalar boson mass eigenstates, two neutral scalars $h^0, H^0$, one neutral pseudoscalar $A^0$, and two charged scalars $H^\pm$. The coupling of all the Higgs bosons to the vector bosons are reduced. The couplings to the fermions are enhanced (up-type) and suppressed (down-type) as $\tan \beta$ increases. At tree level the masses are related by

$$m_{H^\pm}^2 = m_{A^0}^2 + m_W^2,
\quad m_{H^0}^2 + m_{A^0}^2 = m_{A^0}^2 + m_Z^2.$$

(89)

At tree level in SUSY $m_{h^0} \leq M_Z$ however there are large quantum corrections ($m_{h^0} \lesssim 140\text{GeV}$).

## 9 Beyond the Standard Model Physics

As discussed in Section 7 the Standard Model has 18 free parameters, although in principle we should also include the $\Theta$ parameter of QCD. We now need more parameters to incorporate neutrino masses. Despite the excellent description of all current experimental data there are still a number of important questions the Standard Model does not answer.

- What are the values of these parameters?
- Why is the top quark so much heavier than the electron?
- Why is the $\Theta$ parameter so small?
- Is there enough CP-violation to explain why we are here, i.e. the matter-antimatter asymmetry of the universe?
- What about gravity?

While these are all important questions there is no definite answer to any of them.

There are however a large number of models of Beyond the Standard Model (BSM) physics which motivated by trying to address problems in the Standard Model. Given the lack of any experimental evidence of BSM physics the field is driven by theoretical and ascetic arguments, and unfortunately fashion.
All models of BSM physics predict either new particles or differences from the Standard Model, otherwise they cannot be distinguished experimentally from the Standard Model. There are a number of ways of looking for BSM effects:

**Collider Experiments** if the theory contains new particles these should be produced in collider experiments and decay to give Standard Model particles, currently the searches at the energy frontier are at the LHC general-purpose detectors ATLAS and CMS;

**Precision Experiments** measure something predicted by the Standard Model to very high accuracy and compare the results with the theoretical prediction, examples include the LEP/SLD precision measurements at the $Z^0$ pole and the anomalous magnetic moment, $g - 2$, of the muon;

**Rare Decays or Processes** measure the cross section or decay rate for some process which the Standard Model predicts to be small (or zero). Examples include: neutron electric dipole moment experiments, proton decay experiments, neutrino mixing experiments, rare B and kaon decay and CP-violation experiments (BELLE, BaBar, NA48/62, LHCB).

In many ways these approaches are complimentary. Some effects, *e.g.* CP-violation, are best studied by dedicated experiments but if the result of these experiments differs from the SM there should be new particles which are observable at collider experiments.

We will consider the collider signals of BSM physics in detail but only look at the constraints from low-energy physics as we look at various models. The most important low energy constraints are flavour changing neutral currents and proton decay. Often other constraints, *e.g.* from astrophysics and cosmology are also considered.

### 9.1 Models

We will briefly review some of the more promising models and then look at the implications of these models for collider physics taking a pragmatic view looking at the different possible signatures rather than the details of specific models.

There are a wide range of models: grand unified theories; Technicolor; supersymmetry; large extra dimensions; small extra dimensions; little Higgs models; unparticles . . . . Depending on which model builder you talk to they may be almost fanatical in their belief that one of these models is realized in nature.

#### 9.1.1 Grand Unified Theories

The first attempts to answer the problems in the Standard Model were *Grand Unified Theories* (GUTs.) The basic idea is that the Standard Model gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is the subgroup of some larger gauge symmetry. The simplest group is $SU(5)$, which we will consider here, other examples include $SO(10)$. $SU(5)$ has $5^2 - 1 = 24$ generators which means there are 24 gauge bosons. In the Standard Model there are 8 gluons and 4 electroweak gauge bosons ($W^\pm, W^0, B^0 \Rightarrow W^\pm, \gamma, Z^0$). Therefore there are 12 new gauge bosons $X^\pm \frac{2}{3}$ and $Y^\pm \frac{1}{3}$. The right-handed down type quarks and left
handed leptons form a $5$ representation of $SU(5)$. The rest of the particles form a $10$ representation of the gauge group $SU(5)$. The rest of the particles form a $10$ representation of the gauge group $SU(5)$.

\[
\begin{pmatrix}
0 & u^c & -u^c & -u & -d \\
u^c & 0 & u^c & -u & -d \\
u^c & -u^c & 0 & -u & -d \\
0 & u & u & 0 & -e^c \\
d & d & d & e^c & 0
\end{pmatrix}_{L}.
\]  

(90)

In this model there are two stages of symmetry breaking. At the GUT scale the $SU(5)$ symmetry is broken and the $X$ and $Y$ bosons get masses. At the electroweak scale the $SU(2) \times U(1)$ symmetry is broken as before. There are three problems with this theory: the couplings do not unify at the GUT scale; why is the GUT scale higher than the electroweak scale; and proton Decay. We will come back to the first two of these questions.

\[\begin{array}{c}
\text{Proton} \\
\xrightarrow{\text{d}}
\end{array}
\begin{array}{c}
\text{u} \\
\xrightarrow{\text{X}}
\end{array}
\begin{array}{c}
\text{e}^+ \\
\xrightarrow{\text{u}}
\end{array}
\begin{array}{c}
\pi^0 \\
\xrightarrow{\text{d}}
\end{array}
\]

Figure 39: Proton Decay in a Grand Unified theory.

**Proton Decay**  Grand unified theories predict the decay of the proton via the exchange of the $X$ and $Y$ bosons, as shown in Fig. 39. We would expect this decay rate to go like

\[\Gamma(p \to \pi^0 e^+) \sim \frac{M_p^5}{M_X},\]  

(91)

where $M_X$ is the mass of the $X$ boson and $M_p$ the mass of the proton, on dimensional grounds.

There are limits on the proton lifetime from water Čerenkov experiments. The decay of the proton will produce an electron which is travelling faster than the speed of light in water. This will give Čerenkov radiation, just as the electron produced in the weak interaction of a neutrino does. This is used to search for proton decay. As there is no evidence of proton decay there is limit of

\[\tau_p \geq 1.6 \times 10^{32} \text{ years}\]  

(92)

on the proton lifetime. This means $M_X > 10^{16-17}$GeV which is larger than preferred by coupling unification. Proton decay gives important limits on other models.
9.1.2 Hierarchy Problem

The vast majority of new physics models are motivated by considering the hierarchy problem, i.e. why is the electroweak scale so much less than the GUT or Planck (where gravity becomes strong) scales? It is more common to discuss the technical hierarchy problem which is related to the Higgs boson mass. If we look at the Higgs mass there are quantum corrections from fermion loops such as that shown in Fig. 40. This gives a correction to the Higgs mass,

\[
\delta M^2_{H_f} = \frac{|g_f|^2}{4} \int \frac{d^4k}{(2\pi)^4} \frac{\text{tr} \left[ (k+p+m_f)(k+m_f) \right]}{\left[ (k+p)^2 - m_f^2 \right] \left[ k^2 - m_f^2 \right]},
\]

(93)

where \( p \) is the four-momentum of the Higgs boson, \( k \) the four-momentum flowing in the loop, \( g_f \) the coupling of the Higgs boson to the fermion and \( m_f \) the fermion mass. We need to introduce an ultra-violet cut-off, \( \Lambda \), to regularize the integral giving

\[
\delta M^2_{H_f} = \frac{|g_f|^2}{16\pi^2} \left[ -2\Lambda^2 + 6m_f^2 \ln (\Lambda/m_f) \right].
\]

(94)

So either the Higgs mass is the GUT/Planck scale or there is a cancellation

\[
M^2_H = M^2_{H,\text{bare}} + \delta M^2_H,
\]

(95)
of over 30 orders of magnitude to have a light Higgs boson.

This worries a lot of BSM theorists, however there are values of the Higgs boson mass for which the Standard Model could be correct up to the Planck scale. The Higgs boson mass is \( m_H^2 = \lambda v^2 \). There are two constraints on the mass: the coupling should be perturbative, \( \lambda \lesssim 1 \); the vacuum must be non-trivial, \( \lambda \rightarrow 0 \) is forbidden. As can be seen in Fig. 41 there is an island of stability in the middle where the Standard Model can be valid to the Planck scale.

Many solutions to the hierarchy problem have been proposed. They come in and out of fashion and occasionally new ones are proposed. Examples include: Technicolor; supersymmetry; extra dimensions; and little Higgs models.

9.1.3 Technicolor

Technicolor is one of the oldest solutions to the hierarchy problem. The main idea is that as the problems in the theory come from having a fundamental scalar particle they can be solved by not having one. The model postulates a new set of gauge interactions "Technicolor", which acts on new technifermions. We think of this interaction like QCD,
although different gauge groups have been considered. The technifermions form bound states, the lightest being technipions. Using the Higgs mechanism these technipions give the longitudinal components of the $W^\pm$ and $Z$ bosons, and hence generate the gauge boson masses. There must also be a way to generate the fermions masses, *Extended Technicolor*. It has proved hard to construct realistic models which are not already ruled out. For many years Technicolor fell out of fashion, however following the introduction of little Higgs models there has been a resurgence of interest and the new walking Technicolor models look more promising.

### 9.1.4 Supersymmetry

If there is a scalar loop in the Higgs propagator, as shown in Fig. 42. We get a new contribution to the Higgs mass,

$$\delta M_{H\bar{H}}^2 = \frac{\lambda_s}{16\pi^2} \left( \Lambda^2 - 2M_S^2 \ln \left( \frac{\Lambda}{M_S} \right) \right),$$

(96)

where $M_S$ is the mass of the new scalar particle. If there are two scalars for every fermion, with the same mass and $\lambda_s = |g_f|^2$ the quadratic dependence cancels. Theorists like to
Table 5: Particle content of the Minimal Supersymmetric Standard Model.

<table>
<thead>
<tr>
<th>SM particle</th>
<th>Spin</th>
<th>SUSY particle</th>
<th>Spin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron</td>
<td>1/2</td>
<td>Selectron</td>
<td>0</td>
</tr>
<tr>
<td>Neutrino</td>
<td>1/2</td>
<td>Sneutrino</td>
<td>0</td>
</tr>
<tr>
<td>Up</td>
<td>1/2</td>
<td>Sup</td>
<td>0</td>
</tr>
<tr>
<td>Down</td>
<td>1/2</td>
<td>Sdown</td>
<td>0</td>
</tr>
<tr>
<td>Gluon</td>
<td>1</td>
<td>Gluino</td>
<td>1/2</td>
</tr>
<tr>
<td>Photon</td>
<td>1</td>
<td>Photino</td>
<td>1/2</td>
</tr>
<tr>
<td>Z</td>
<td>1</td>
<td>Zino</td>
<td>1/2 Neutralinos</td>
</tr>
<tr>
<td>Higgs</td>
<td>0</td>
<td>Higgsino</td>
<td>1/2</td>
</tr>
<tr>
<td>W^+</td>
<td>1</td>
<td>Wino</td>
<td>1/2 Charginos</td>
</tr>
<tr>
<td>H^+</td>
<td>0</td>
<td>Higgsino</td>
<td>1/2</td>
</tr>
</tbody>
</table>

have symmetries to explain cancellations like this, *Supersymmetry* (SUSY). For every fermionic degree of freedom there is a corresponding bosonic degree of freedom: all the SM fermions have two spin-0 partners; all the SM gauge bosons have a spin-\( \frac{1}{2} \) partner. The full particle content of the theory is given in Table 5. In SUSY models we need to have two Higgs doublets to give mass to both the up- and down-type quarks in a way which is invariant under the supersymmetric transformations.

There are major two reasons, in addition to the solution of the hierarchy problem, to favour SUSY as an extension of the SM.

**Coleman-Mandula theorem** If we consider any extension to the Poincaré group any new generators which transform as bosons lead to a trivial S-matrix, i.e. scattering only through discrete angles. Later Haag, Lopuszanski and Sohnius showed that SUSY is the only possible extension of the Poincaré group which doesn’t give a trivial S-matrix.

**SUSY coupling unification** In SUSY GUTS the additional SUSY particles change the running of the couplings and allow the couplings to truly unify at the GUT scale, as shown in Fig. 43. However, with increasingly accurate experimental measurements of the strong coupling this is no longer quite true.

In the modern view of particle physics we construct a theory by specifying the particle content and symmetries. All the terms allowed by the symmetries are then included in the Lagrangian. If we do this in supersymmetric models we naturally get terms which do not conserve lepton and baryon number. This leads to proton decay as shown in Fig. 44. Proton decay requires that both lepton and baryon number conservation are violated. The limits on the proton lifetime lead to very stringent limits on the product of the couplings leading to proton decay.

\[
\lambda'_{11k} \cdot \lambda''_{11k} \lesssim 2 \cdot 10^{-27}.
\]  

(97)

Only natural way for this to happen is if some symmetry requires that one or both couplings are zero. Normally a multiplicatively conserved symmetry *R-parity*

\[
R_p = (-1)^{3B + L + 2S},
\]  

(98)
Flavour Changing Neutral Currents In the Standard Model the only interactions which change the quark flavour are those with the $W^\pm$ boson. So any processes which change the flavour of the quarks, but not the charge, \textit{Flavour Changing Neutral Currents} (FCNCs), must be loop mediated.

There are two important types: those which change the quark flavour with the emission of a photon, \textit{i.e.} $b \to s\gamma$; those which give meson-antimeson mixing, \textit{e.g.} $B - \bar{B}$ mixing.
Both are important in the Standard Model and in constraining possible new physics models.

In the Standard Model flavour changing neutral currents are suppressed by the Glashow-Iliopoulos-Maiani (GIM) mechanism. If we consider neutral Kaon mixing, as shown in Fig. 45, and the rare Kaon decays $K^0_L \rightarrow \mu^+\mu^-$ and $K^0_L \rightarrow \gamma\gamma$, as shown in Fig. 46.

Considering only two generations for simplicity all these diagrams go like

$$\Gamma(K^0_L \rightarrow \mu^+\mu^-) \sim 2 \times 10^{-5} \Gamma(K^0_L \rightarrow \gamma\gamma)$$

This suppression is called the GIM mechanism and explains why

$$\frac{1}{M^2} \frac{m_u^2 - m_c^2}{M^2}$$

times a factor due to the Cabibbo mixing angle where $M$ is the largest mass left after the removal of one W propagator, i.e. $M_W$ for $K^0 - \bar{K}^0$ mixing and $K^0_L \rightarrow \mu^+\mu^-$, and $m_e$ for $K^0_L \rightarrow \gamma\gamma$. This suppression is called the GIM mechanism and explains why

The current experimental results are in good agreement with the SM. This often proves a problem in BSM physics as there are often new sources of FCNCs.

In SUSY theories the SUSY partners also give contributions to FCNCs, as shown in Fig. 47. In this case the diagrams proportional to the mass difference of the squarks.

Provide the SUSY breaking masses are flavour independent this is not a problem, as the mass differences are the same as the SM. It is also not a problem if there is no flavour mixing in the model. In general both these things are possible and must be considered.

**SUSY Breaking** What are the 120 SUSY breaking parameters? In general there are: SUSY breaking masses for the scalars; SUSY breaking masses for the gauginos; $A$ terms which mix three scalars; mixing angles and CP-violating phases. We need a model of
where these parameters come from in order to do any phenomenological or experimental studies. We therefore use models which predict these parameters from physics at higher energy scales, i.e. the GUT or Planck scale. In all these models SUSY is broken in a hidden sector. The models differ in how this SUSY breaking is transmitted to the visible sector, i.e. the MSSM particles.

**SUGRA** SUSY breaking is transmitted via gravity. All the scalar ($M_0$) and gaugino ($M_{1/2}$) masses are unified at the GUT scale. The $A$ and $B$ terms are also universal. The known value of $M_Z$ is used to constrain the $\mu$ and $B$ parameters leaving $\tan\beta = v_1/v_2$ as a free parameter. There are five parameters which give the mass spectrum: $M_0$, $M_{1/2}$, $\tan\beta$, $\text{sgn}\mu$, $A$. The gluino mass is correlated with $M_{1/2}$ and slepton mass with $M_0$.

**GMSB** In gauge mediated SUSY breaking (GMSB) the flavour-changing neutral current problem is solved by using gauge fields instead to gravity to transmit the SUSY breaking. The messenger particles, $X$, transmit the SUSY breaking. The simplest choice is a complete $SU(5)$ 5 or 10 of particles transmitting the SUSY breaking to preserve the GUT symmetry. The fundamental SUSY breaking scale $\lesssim 10^{10}\text{GeV}$ is lower than in gravity mediated models. The gaugino masses occur at one-loop, $M_\tilde{g} \sim \alpha_s N_X \Lambda$ while the scalar masses occur at two-loop, $M_{\tilde{q}} \sim \alpha_s^2 \sqrt{N_X} \Lambda$, where $\Lambda$ is the breaking scale and $N_X$ the number of messenger fields. The true LSP is the almost massless gravitino. The lightest superpartner is unstable and decays to gravitino and can be neutral, e.g. $\tilde{\chi}_1^0$, or charged, e.g. $\tilde{\tau}_1$.

**AMSB** The superconformal anomaly is always present and can give anomaly mediated SUSY breaking (AMSB). This predicts the sparticle masses in terms of the gravitino mass, $M_{3/2}$. The simplest version of the model predicts tachyonic particles so another SUSY breaking mechanism is required to get a realistic spectrum, e.g. adding universal scalar masses ($M_0$). The model has four parameters $M_0$, $M_{3/2}$, $\tan\beta$ and $\text{sgn}\mu$. In this model the lightest chargino is almost degenerate with the lightest neutralino.
The mass spectrum in the models is different, as shown in Fig. 48. The main differences are: the mass splitting between gluino and electroweak gauginos; the mass splitting of the squarks and sleptons; and the nature of the LSP.

**Muon g-2** Another important low energy constraint on BSM physics is the anomalous magnetic moment of the muon. The magnetic moment of any fundamental fermion is

\[ \mu = g \left( \frac{e}{2m} \right) S, \]

where \( g \) is the \( g \)-factor, \( m \) the mass and \( S \) the spin of the particle. The Dirac equation predicts \( g = 2 \). However there are quantum corrections, as shown in Fig. 49, which lead to an anomalous magnetic moment, \( g - 2 \).

![Figure 49: Vertex correction contributing to the anomalous muon magnetic moment in the Standard Model.](image)

There are also quark loops in the photon propagator, as shown in Fig. 50. This is a low energy process so we can not use perturbative QCD. Instead we must use the measured \( e^+e^- \) total cross section and the optical theorem to obtain the corrections which leads to an experimental error on the theoretical prediction. In many BSM theories, for example in SUSY, there are additional corrections from diagrams, such as that shown in Fig. 51.

![Figure 50: Quark loop in the photon propagator which contributes to the anomalous muon magnetic moment in the Standard Model.](image)

![Figure 51: Example of a SUSY correction to the muon magnetic moment.](image)

The original experimental result disagreed with the SM at 2.6\( \sigma \), but there was an error in the sign in one of the terms in the theoretical calculation reducing the significance to about 1.4\( \sigma \). However if you measure enough quantities some of them should disagree with the prediction by more the 1 sigma (about 1/3), and some by 2 sigma (4.6%) or 3 sigma (0.3%). This is why we define a discovery to be 5 sigma (6 \( \times \) 10^{-5}%), so this is nothing to worry about.

173
Rare $B$ decays  There is an amazing consistency of the current flavour physics measurements. However, many new physics models can have a similar pattern in their flavour sector, the new physics model must have this otherwise it is experimentally excluded. However, there can still be new physics in rare processes (like $B^+ \to \tau^+ \nu_{\tau}$) and CP-asymmetries. One promising examples is the decay $B_s \to \mu^+ \mu^-$. There are two Standard Model contributions from box and penguin diagrams as shown in Fig. 52. Both of these are suppressed by $V_{tb}V_{ts}^*$ giving a Standard Model branching ratio

$$BR_{B_s \to \mu^+ \mu^-}^{(SM)} \approx 10^{-9}.$$  

(101)

This gives a simple leptonic final state with minor theoretical uncertainties but a huge background so the mass resolution is paramount, the expected mass resolution for the LHC experiments is given in Table 6.

<table>
<thead>
<tr>
<th>Exp.</th>
<th>ATLAS</th>
<th>CMS</th>
<th>LHCb</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_m$ (MeV)</td>
<td>77</td>
<td>36</td>
<td>18</td>
</tr>
</tbody>
</table>

Table 6: Expected mass resolution for $B_s \to \mu^+ \mu^-$.

In the MSSM, however, the amplitude involves three powers of $\tan^2 \beta$, so that

$$BR_{B_s \to \mu^+ \mu^-}^{(MSSM)} \propto \tan^6 \beta,$$

(102)

which leads to an enhancement over the SM value by up to three orders of magnitude.

9.1.5 Extra Dimensions

Many theorists believe there are more than 4 dimensions, for example string theories can only exist in 10/11 dimensions. The hierarchy problem can be solved (redefined?) in these models in one of two ways.

1. There is a large extra dimension with size $\sim 1$mm. In this case

$$M_{\text{Planck}}^2 \sim M^{n+2} R^n,$$

(103)

where $M_{\text{Planck}}$ is the observed Planck mass, $M$ is the extra-dimensional Planck mass and $R$ the radius of the additional $n$ dimensions. In this case the Planck mass is of order 1 TeV so there is no hierarchy problem. However the hierarchy in the sizes of the dimensions must be explained.
2. Small extra dimensions in which case the extra dimension is warped. The model has two branes, we live on one and the other is at the Plank scale. The Higgs VEV is suppressed by a warp factor, \( \exp\left(-kr_c\pi\right) \), where \( r_c \) is the compactification radius of the extra dimension, and \( k \) a scale of the order of the Planck scale.

We can consider what happens in extra-dimensional models by studying a scalar field in 5-dimensions. In this case the equation of motion for the scalar field is

\[
\left( \frac{\partial^2}{\partial t^2} - \nabla^2_5 + m^2 \right) \Phi(x, y, z, x_5, t) = 0,
\]

where

\[
\nabla^2_5 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x_5^2}
\]

is the 5-dimensional Laplace operator. If the 5-th dimension is circular we can Fourier decompose the field,

\[
\Phi(x, y, z, x_5, t) = \sum_n \Phi_n(x, y, z, t) \exp(inx_5/R).
\]

The equation of motion therefore becomes,

\[
\sum_n \left( \frac{\partial^2}{\partial t^2} - \nabla^2_4 + m^2 + \frac{n^2}{R^2} \right) \Phi_n(x, y, z, t).
\]

This gives a Kaluza-Klein (KK) tower of states with mass splitting \( \sim 1/R \). There are a number of different models.

**Large Extra Dimensions** Only gravity propagates in the bulk, \( i.e. \) in the extra dimensions. We therefore only get Kaluza-Klein excitations of the graviton. In large extra dimensional models the mass splitting between the KK excitations is small and all the gravitons contribute to a given process. Phenomenologically there are deviations from the SM prediction for SM processes.

**Small Extra Dimensions** Again only gravity propagates in the bulk so there are only KK excitations of the graviton. In this case the mass splitting is large leading to resonant graviton production.

**Universal Extra Dimensions** Another alternative is to let all the Standard Model fields propagate in the bulk, *Universal Extra Dimensions* (UED). All the particles have Kaluza-Klein excitations. It is possible to have a Kaluza-Klein parity, like R-parity in SUSY. The most studied model has one extra dimension and a similar particle content to SUSY, apart from the spins. There are also some 6-dimensional models.
9.1.6 Little Higgs Models

In little Higgs models the Higgs fields are Goldstone bosons associated with breaking a global symmetry at a high scale, $A_S$. The Higgs fields acquire a mass and become pseudo-Goldstone bosons via symmetry breaking at the electroweak scale. The Higgs fields remain light as they are protected by the approximate global symmetry. The model has heavy partners for the photon, $Z^0$, $W^\pm$ bosons and the top quark as well as extra Higgs bosons. The non-linear $\sigma$-model used for the high energy theory is similar to the low energy effective theory of pions which can be used to describe QCD, or in Technicolor models. This similarity with Technicolor models is one of the reasons for the resurgence of Technicolor models in recent years.

The original Little Higgs models had problems with electroweak constraints. The solution is to introduce a discrete symmetry called T-parity, analogous to R-parity in SUSY models. This solves the problems with the precision electroweak data and provides a possible dark matter candidate. This model has a much large particle content than the original Little Higgs model and is more SUSY-like with a partner for each Standard Model particle.

9.1.7 Unparticles

In these models a new sector at a high energy scale with a non-trivial infrared (IR) fixed point is introduced. This sector interacts with the Standard Model via the exchange of particles with a large mass scale leading to an effective theory

$$\frac{C_U A_U^{d_{BZ} - d_U}}{M_U^k} O_{SM} O_U,$$

where: $d_U$ is the scaling dimension of the unparticle operator $O_U$; $M_U$ is the mass scale for the exchanged particles; $O_{SM}$ is the Standard Model operator; $d_{BZ}$ is the dimension of the operator in the high energy theory; $k$ gives the correct overall dimension of the interaction term. This leads to new operators which give deviations from the Standard Model predictions for various observables.
Figure 54: Jet $p_{\perp}$ spectrum for various numbers of extra dimensions in the ADD model taken from Ref. [25].

9.2 Beyond the Standard Model Signatures

Before we go on and consider the signals of models of new physics in great detail it is worthwhile considering what we expect to see in general. Most models of new physics predict either the existence of more particles than the Standard Model or new operators which give deviations from the Standard Model predictions. The signatures of the model depend on either how these particles are produced and decay or the type of deviations expected. In any study of BSM physics the most important thing is to understand the Standard Model backgrounds. Often the signal is at the tail of some distribution and the limits of our ability to calculate or simulate it.

9.2.1 Deviations from the Standard Model

There can be deviations from what is expected in the Standard Model due to: compositeness; exchanging towers of Kaluza-Klein gravitons in large extra dimension models; unparticle exchange; .... This tends to give changes in the shapes of spectra. Therefore in order to see a difference you need to know the shape of the Standard Model prediction.

Example I: High $p_{\perp}$ jets One possible signal of compositeness is the production of high $p_{\perp}$ jets. At one point there was a disagreement between theory and experiment at the Tevatron. However, this was not due to new physics but too little high-$x$ gluon in the PDFs. Now as well as looking in the $p_{\perp}$ spectra at central rapidities where we expect to see a signal of BSM physics we also look at high rapidity as a disagreement at both central and high rapidities is more likely to be due to the parton distribution functions. An example of the jet $p_{\perp}$ spectrum at a range of rapidities is shown in Fig. 23.

Example II: Unparticles Many models predict deviations in the Drell-Yan mass spectra, for example in an unparticle model with the exchange of virtual spin-1 unparticles, see Fig. 53. However, we need to be careful as higher order weak corrections which can also change the shape are often neglected.
Example III: PDF uncertainty or new physics  In the ADD model of large extra dimensions there are changes in the shape of the jet $p_{\perp}$ and dijet mass spectra due to the exchange of KK towers of gravitons and their destructive interference with SM, as shown in Fig. 54.

9.2.2 Monojets

There are a range of models which predict monojet signals with the production of a quark or gluon which is recoiling against either: a stable neutral particle; a tower of KK gravitons in large extra dimension models; unparticles; . . . .

Example IV: Mono-jets at the SppS  In Ref. [26] the UA1 collaboration reported: 5 events with $E_{\perp,\text{miss}} > 40$ GeV and a narrow jet; 2 events with $E_{\perp,\text{miss}} > 40$ GeV and a neutral EM cluster. They could “not find a Standard Model explanation”, and compared their findings with a calculation of SUSY pair-production [27]. They deduced a gluino mass larger than around 40 GeV. In Ref. [28], the UA2 collaboration describes similar events, also after 113 nb$^{-1}$, without indicating any interpretation as strongly as UA1. In Ref. [29] S. Ellis, R. Kleiss, and J. Stirling calculated the backgrounds to that process more carefully, and showed agreement with the Standard Model.

There are many different Standard Model electroweak backgrounds and a careful comparison shows they are currently in agreement with the Standard Model, see Fig. 55.

9.2.3 New Particle Production

In general there are two cases for models in which new particles are produced.

1. The model has only a few new particles, mainly produced as s-channel resonances. Examples include: Z-prime models; little Higgs models; small extra dimension models, . . . .

2. The model has a large number of new particles. Examples include: SUSY; UED; little Higgs models with T-parity, . . . .
In the first type of model the main signal is the production of s-channel resonances while in the second class of models the signals are more varied and complex.

### 9.2.4 Resonance Production

The easiest and cleanest signal in hadron collisions is the production of an s-channel resonance which decays to $e^+e^-$ or $\mu^+\mu^-$. Resonances in this and other channels are possible in: Little Higgs models; $Z'$ models; UED; Small Extra Dimensions. Backgrounds can be remove using sideband subtraction.

#### Example V: Resonant Graviton Production

The best channel, $e^+e^-$, gives a reach of order 2 TeV depending on the cross section for the LHC running at $\sqrt{s} = 14$ GeV. Other channels $\mu^+\mu^-$, $gg$, and $W^+W^-$ are possible. If the graviton is light enough the angular distribution of the decay products can be used to measure the spin of the resonance. An example of the dilepton mass spectrum in this model is shown in Fig. 56.

A lot of models predict hadronic resonances. This is much more problematic due to the mass resolution which smears out narrow resonances and the often huge QCD backgrounds. Although background subtraction can be used the ratio of the signal to background is often tiny, for example Fig. 57 shows the measured $Z \rightarrow b\bar{b}$ peak at the Tevatron. 57
Figure 57: Dijet mass spectrum for bottom quark jets at the Tevatron taken from Ref. [31].

9.2.5 SUSY-like models

Most of the other models are “SUSY”-like, i.e. they contain: a partner of some kind for every Standard Model particle; often some additional particles such as extra Higgs bosons; a lightest new particle which is stable and a dark matter candidate.

A lot of new particles should be produced in these models. While some particles may be stable, the the majority of these particles decay to Standard Model particles. Therefore we expect to see: charged leptons; missing transverse energy from stable neutral particles or neutrinos; jets from quarks, perhaps with bottom and charm quarks; tau leptons; Higgs boson production; photons; stable charged particles. It is worth noting that seeing an excess of these does not necessarily tell us which model has been observed.

The archetypal model containing large numbers of new particles which may be accessible at the LHC is SUSY. Other models are UED and the Little Higgs Model with T-parity. However, in practice UED is mainly used as a straw-man model for studies trying to show that a potential excess is SUSY.

Two statements which are commonly made are: the LHC will discover the Higgs boson; the LHC will discover low-energy SUSY if it exists. The first is almost certainly true, however the second is only partially true.

In hadron collisions the strongly interacting particles are dominantly produced. Therefore in SUSY squark and gluino production has the highest cross section, for example via the processes shown in Fig. 58.

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8 *i.e.* the decay length of the particle is such that the majority of the particles escape from the detector before decaying. In practice this happens for lifetimes greater than $10^{-7}$ s.
These particles then decay in a number of ways. Some of them have strong decays to other strongly interacting SUSY particles, for example via the processes shown in Fig. 59. However the lightest strongly interaction SUSY particle, squark or gluino, can only decay weakly, as shown in Fig. 60. The gluino can only have weak decays with virtual squarks or via loop diagrams. This is the main production mechanism for the weakly interacting SUSY particles.

The decays of the squarks and gluinos will produce lots of quarks and antiquarks. The weakly interacting SUSY particles will then decay giving more quarks and leptons. Eventually the lightest SUSY particle which is stable will be produced. This behaves like a neutrino and gives missing transverse energy. So the signal for SUSY is large numbers of jets and leptons with missing transverse energy. This could however be the signal for many models containing new heavy particles.
Figure 61: Expected limits in SUSY parameter space for searches using jets and missing transverse energy and jets, leptons and missing transverse energy for the LHC running at $\sqrt{s} = 14$ TeV taken from Ref. [32].

Figure 62: Expected limits in SUSY parameter space for searches using jets, leptons and missing transverse energy for the LHC running at $\sqrt{s} = 14$ TeV taken from Ref. [32].

All SUSY studies fall into two categories: search studies which are designed to show SUSY can be discovered by looking for a inclusive signatures and counting events; measurement studies which are designed to show that some parameters of the model, usually masses, can be measured.

There is a large reach looking for a number of high transverse momentum jets and leptons, and missing transverse energy, see Figs. 61 and 62. It is also possible to have the production of the $Z^0$ and Higgs bosons and top quarks. In many cases the tau lepton may be produced more often than electrons or muons.

Once we observe a signal of SUSY there are various approaches to determine the properties of the model. The simplest of these is the effective mass

$$M_{\text{eff}} = \sum_{i=1}^{n} p_{T,i}^\text{jet} + \not{E}_T,$$

(109)
which is strongly correlated with the mass of strongly interacting SUSY particles and can be used to measure the squark/gluino mass to about 15%, see Fig. 63.

The analyzes we have just looked at are those that are used to claim the LHC will discover SUSY but this is not really what they tell us. They don’t really discover SUSY. What they see is the production of massive strongly interacting particles, this does not have to be SUSY, it could easily be something else. In order to claim that a signal is SUSY we would need to know more about it. SUSY analyzes tend to proceed by looking for characteristic decay chains and using these to measure the masses of the SUSY particles and determine more properties of the model.

Given most of the searches are essentially counting experiments it is important to understand the Standard Model backgrounds which can be challenging, see Fig. 64.

9.2.6 Model Independent Searches

A popular approach in recent years has been to use experimental data to place constraints on general signatures of new physics parametrised by additional operators suppressed by a scale $\Lambda$,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SM}} + \sum_i \frac{1}{\Lambda^{d_i-4}} c_i \mathcal{O}_i.$$  

(110)
where $O_i$ are a set of dimension $d_i$ operators with Wilson coefficients $c_i$. The Wilson coefficients $c_i = c_i(\mu_R)$ run as functions of the renormalisation group scale $\mu_R$.

**Effective field theories:** Even if we restrict ourselves to dimension-6 operators in our expansion the total number of Wilson coefficients is extremely large. Since we are only interested in operators that contribute to on-shell observables we can reduce to a set of independent operators by systematically using the equations of motion for each field and integration-by-parts identities. Nevertheless the number of independent dimension-6 operators for $SU(3) \times SU(2)_L \times U(1)_Y$ with 3 fermion generations is 3045 [34]. Constraining the Wilson coefficients will be a major challenge even with the large amount of new data coming from the LHC. In addition the expansion in the scale $\Lambda$ must be carefully compared to the energy scales present in the observable. The EFT expansion will only be valid if $\Lambda \gg Q^2$ so that it may not be applicable to events with large transverse energy - which would hope to be of greatest sensitivity to new phenomena.

**Simplified Models:** While not completely model independent, identifying a limited set of additional operators and interactions that can parameterise potential new physics in particular observables is a useful technique. This intermediate point between a complete EFT and a specific model reduces the number of free parameters to a manageable level and at the same time retaining sensitivity to large classes of theories. Some further details and additional references can be found in [35].

### A Kinematics and Cross Sections

#### A.1 Kinematics

The basic language of all phenomenology is that of relativistic kinematics, in particular four-vectors. In hadron collisions because we do not know what fraction of the beam
momenta is transferred to the partonic system it is preferable to use quantities, such as the transverse momentum, \( p_\perp \), with respect to the beam direction which are invariant under longitudinal boosts along the beam direction to describe the kinematics. In addition to the transverse momentum we use the rapidity, \( y \), and massless pseudorapidity, \( \eta \),

\[
y = \frac{1}{2} \ln \frac{E + p_\perp \text{massless}}{E - p_\perp} \quad \eta = -\ln \tan \frac{\theta}{2},
\]

because rapidity differences are invariant under longitudinal boosts. Particles with small rapidities are produced at an angle close to 90° degrees to the beam direction while particles with large positive (negative) rapidities are travelling in the forward (backward) beam direction. The pseudorapidity is more often used experimentally as it is related to the measured scattering angle.

The four-momentum can be written as

\[
p^\mu = (E, p_x, p_y, p_z) = (m_\perp \cosh y, p_\perp \cos \phi, p_\perp \sin \phi, m_\perp \sinh y),
\]

where \( m_\perp^2 = p_\perp^2 + m^2 \). The one-particle phase-space element can also be rewritten in terms of \( y \) and \( p_\perp \) as

\[
\frac{d^4p}{(2\pi)^4} \delta(p^2 - m^2) \theta(E) = \frac{d^3p}{(2\pi)^2 2E} = \frac{dy dp_\perp}{2(2\pi)^3}.
\]

### A.2 Cross Sections

The starting point of all collider physics calculations is the calculation of the scattering cross section. The cross section for a \( 2 \to n \) scattering processes, \( a + b \to 1 \ldots n \), is

\[
d\sigma = \frac{(2\pi)^4}{4 \sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}} d\Phi_n(p_a + p_b; p_1 \ldots p_n)|\mathcal{M}|^2,
\]

where \( p_{a,b} \) and \( p_{i=1,\ldots,n} \) are the momenta of the incoming and outgoing particles, respectively. The matrix element squared \(|\mathcal{M}|^2\) is summed/averaged over the spins and colours of the outgoing/incoming particles. The \( n \)-particle phase-space element is

\[
d\Phi_n(p_a + p_b; p_1 \ldots p_n) = \delta^4\left(p_a + p_b - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \left(\frac{d^3p_i}{(2\pi)^32E_i}\right),
\]

where \( E_i \) is the energy of the \( i \)-th particle. It is conventional to define \( s = (p_a + p_b)^2 \). For massless incoming particles \( 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2} = 2s \).

Although modern theoretical calculations involve ever higher multiplicity final states in these lectures we will primarily deal with \( 2 \to 2 \) scattering processes in which case

\[
d\Phi_2(p_a + p_b; p_1, p_2) = \delta^4(p_a + p_b - p_1 - p_2) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2},
\]

\[
= \delta(E_a + E_b - E_1 - E_2) \frac{1}{(2\pi)^6 4E_1E_2} |p_1|^2 |p_2|^2 d|p_1| |p_2| d\cos \theta d\phi,
\]

\[
= \frac{1}{8\pi(2\pi)^4} \frac{1}{\sqrt{s}} |p_1| d\cos \theta,
\]
where \( |p_1| \) is the magnitude of the three-momenta of either of the outgoing particles and \( \theta \) and \( \phi \) are the polar and azimuthal scattering angles, respectively. The cross section

\[
d\sigma = \frac{1}{16\pi s} \frac{|p_1|}{\sqrt{s}} d\cos \theta |\mathcal{M}|^2.
\]

In is conventional to describe the scattering process in terms of the Mandelstam variables

\[
s = (p_a + p_b)^2,
\]

\[
t = (p_a - p_1)^2,
\]

\[
u = (p_a - p_2)^2.
\]

There are only two independent Mandelstam variables

\[
s + t + u = m_1^2 + m_2^2 + m_a^2 + m_b^2 \rightarrow 0.
\]

In terms of these variables

\[
d\sigma = \frac{1}{16\pi s^2} dt |\mathcal{M}|^2.
\]

### A.3 Cross Sections in Hadron Collisions

In hadron collisions there is an additional complication as the partons inside the hadrons interact. The hadron–hadron cross section is

\[
d\sigma_{AB} = \sum_{ab} \int_0^1 dx_1 dx_2 f_{a/A}(x_1, \mu_F^2) f_{b/B}(x_2, \mu_R^2) \hat{\sigma}_{ab} (\hat{s}, \mu_F^2, \mu_R^2),
\]

where \( x_{1,2} \) are momentum fractions of the interacting partons with respect to the incoming hadrons, \( \hat{s} = x_1 x_2 \hat{s} \), \( \hat{\sigma}_{ab}(\hat{s}, \mu_F^2, \mu_R^2) \) is the parton-level cross section for the partons \( a \) and \( b \) to produce the relevant final state, \( f_{a/A}(x, \mu_F^2) \) is the parton distribution function (PDF) giving the probability of finding the parton \( a \) in the hadron \( A \), and similarly for \( f_{b/B}(x, \mu_R^2) \). The factorization and renormalisation scales are \( \mu_F \) and \( \mu_R \), respectively.

In hadron collisions we usually denote the variables for partonic process with \( \hat{\cdot} \), e.g. \( \hat{s}, \hat{t} \) and \( \hat{u} \) for the Mandelstam variables.

#### A.3.1 Resonance production (2 \( \rightarrow \) 1 processes)

The simplest example of a hadronic cross section is the production of an s-channel resonance, for example the \( Z^0 \) or Higgs bosons. We assume that the incoming partons are massless so that the 4-momenta of the incoming partons are:

\[
p_{a,b} = x_{1,2}(E, 0, 0, \pm E),
\]

where \( E \) is beam energy in the hadron–hadron centre-of-mass system of collider such that \( s = 4E^2 \). The Breit-Wigner cross section, e.g. for \( Z \) production, is

\[
\hat{\sigma}_{q\bar{q} \rightarrow Z^0 \rightarrow \mu^+ \mu^-} = \frac{1}{N_c^2 M_Z^2} \frac{12\pi \hat{s}}{(\hat{s} - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \Gamma_{q\bar{q} \rightarrow \mu^+ \mu^-}.
\]

In the limit that the width is a lot less than the mass

\[
\frac{1}{(\hat{s} - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \approx \frac{\pi}{M_Z \Gamma_Z} \delta(\hat{s} - M_Z^2),
\]

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the narrow width limit. In this case the partonic centre-of-mass system is constrained to have \( \hat{s} = M_Z^2 \). The rapidity \( \hat{y} \) of the partonic system and \( \hat{s} \) are related to the momentum fractions \( x_{1,2} \) by

\[
\hat{s} = x_1 x_2, \quad \text{and} \quad \hat{y} = \frac{1}{2} \ln \frac{x_1 + x_2 + x_1 - x_2}{x_1 + x_2 - x_1 + x_2} = \frac{1}{2} \ln \frac{x_1}{x_2}.
\]

(125)

Inverting these relationships we obtain

\[
x_{1,2} = \frac{1}{\sqrt{\hat{s}}} \frac{e^{\pm \hat{y}}}{s} \quad \text{and} \quad \hat{y} = \frac{1}{2} \ln \frac{x_1^2 s}{\hat{s}} \leq \ln \frac{2E}{\sqrt{s}} = \hat{y}_{\text{max}}.
\]

(126)

This allows us to change the variables in the integration using

\[
s dx_1 dx_2 = d\hat{s} d\hat{y},
\]

(127)

giving the differential cross section

\[
\frac{d\sigma_{AB \to Z^0 \to \mu^+ \mu^-}}{d\hat{y}} = \sum_{a,b=q\bar{q}} x_1 f_q/A(x_1, \mu_F^2) x_2 f_{\bar{q}}/B(x_2, \mu_F^2) \frac{12\pi^2}{N_c^2 M_Z^4} \Gamma_{qq} B_{\mu^+ \mu^-}.
\]

(128)

A.3.2 2 \to 2 Scattering Processes

For most 2 \to 2 scattering processes in hadron–hadron collisions it is easier to work in terms of the rapidities \( y_3, y_4 \) and transverse momentum, \( p_\perp \), of the particles. We introduce average (centre-of-mass) rapidity and rapidity difference,

\[
\bar{y} = \frac{y_3 + y_4}{2} \quad \text{and} \quad y^* = \frac{y_3 - y_4}{2},
\]

(129)

which are related to the Bjorken \( x \) values by

\[
x_{1,2} = \frac{p_\perp}{\sqrt{2}} \left( e^{\pm y_3} + e^{\pm y_4} \right) = \frac{p_\perp}{2\sqrt{s}} e^{\pm \hat{y}} \cosh y^*.
\]

(130)

Therefore

\[
\hat{s} = M_{12}^2 = 4p_\perp^2 \cosh y^* \quad \text{and} \quad \hat{t}, \hat{u} = -\frac{\hat{s}}{2} (1 + \tanh y^*).
\]

The partonic cross section, assuming all the particles are massless, is

\[
\hat{\sigma}_{ab \to 12} = \frac{1}{2s} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} |\mathcal{M}_{ab \to 12}|^2 (2\pi)^4 \delta^4(p_a + p_b - p_1 - p_2),
\]

(131)

Therefore once we include the PDFs, sum over \( a, b \), and integrate over \( x_{1,2} \) the hadronic cross section is

\[
\sigma_{AB \to 12} = \sum_{ab} \int d\hat{y}_1 d\hat{y}_2 d^2p_\perp \frac{f_a(x_1, \mu_F) f_b(x_2, \mu_F)}{16\pi^2 s^2} |\mathcal{M}_{ab \to 12}|^2,
\]

including the factor \( 1/(1 + \delta_{12}) \) for identical final-state particles.
B Flavour Physics

While most of the interactions in the Standard Model preserve the flavour of quarks and leptons the interaction of fermions with the W boson can change the flavour of the quarks and violate CP-conservation.

In order to understand the interactions of the quarks with the W boson we first need to consider the generation of quark masses in the Standard Model. The masses of the quarks come from the Yukawa interaction with the Higgs field

\[
\mathcal{L} = -Y^u Q_L^i \phi d_R^j - Y^d Q_L^i \epsilon^* u_R^j + \text{h.c.},
\]

where \(Y^{u,d}\) are complex \(3 \times 3\) matrices, \(\phi\) is the Higgs field, \(i, j\) are generation indices, \(Q_L^i\) are the left-handed quark doublets and, \(d_R^j\) and \(u_R^j\) are the right down- and up-type quark singlets. When the Higgs field acquires a vacuum expectation value \(\langle \phi \rangle = (0, \frac{v}{\sqrt{2}})\) we get the mass terms for the quarks.

The physical states come from diagonalizing \(Y^{u,d}\) using 4 unitary \(3 \times 3\) matrices, \(V_{L,R}^{u,d}\)

\[
M^f_{\text{diag}} = V_L^f Y^f V_R^{f\dagger} \frac{v}{\sqrt{2}}.
\]

The interaction of the \(W^\pm\) and the quarks is given by

\[
\mathcal{L}_W = -\frac{g}{\sqrt{2}} \left[ d_L^i \gamma^\mu W^-_\mu u_L^i + \bar{u}_L^i \gamma^\mu W^+_\mu d_L^i \right].
\]

The interaction with the mass eigenstates, \(f^M_L = V_L^f f_L^i\), is

\[
\mathcal{L}_W = -\frac{g}{\sqrt{2}} \left[ d_L^i \gamma^\mu W^-_\mu V_{\text{CKM}}^M u_L^i + \bar{u}_L^i \gamma^\mu W^+_\mu V_{\text{CKM}}^M d_L^i \right],
\]

where the Cabibbo-Kobayashi-Maskawa (CKM) matrix

\[
V_{\text{CKM}} = V_L^u C V_L^d = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix},
\]

is a \(3 \times 3\) unitary matrix.

The CKM matrix can be parameterized in terms of three mixing angles, \((\theta_{12}, \theta_{13}, \theta_{23})\) and one phase, \(\delta\),

\[
V_{\text{CKM}} = \begin{pmatrix}
c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
-s_{12} c_23 & -s_{12} s_23 s_{13} e^{i\delta} & c_23 s_{13} \\
s_{12} s_23 & -s_{12} s_23 s_{13} e^{-i\delta} & c_{23} c_{13}
\end{pmatrix},
\]

where \(s_{ij} = \sin \theta_{ij}\) and \(c_{ij} = \cos \theta_{ij}\). As experimentally \(s_{13} \ll s_{23} \ll s_{12} \ll 1\) it is convenient to use the Wolfenstein parameterization: \(s_{12} = \lambda; s_{23} = A \lambda^2\); and \(s_{13} e^{i\delta} = A \lambda^3 (\rho + i \eta)\).

In which

\[
V_{\text{CKM}} = \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^3 (\rho - i \eta) \\
-\lambda & 1 - \frac{1}{2} \lambda^2 & A \lambda^2 \\
A \lambda^3 (1 - \rho - i \eta) & -A \lambda^2 & 1
\end{pmatrix} + \mathcal{O}(\lambda^4).
\]
If we assume that the neutrinos are massless there is no mixing for leptons. We now know that the neutrinos have small masses so there is mixing in the lepton sector. The analogy of the CKM matrix is the Maki-Nakagawa-Sakata (MNS) matrix $U_{\text{MNS}}$.

A number of unitarity triangles can be constructed using the properties of the CKM matrix. The most useful one is

$$V_{ud}V_{ub} + V_{cd}V_{cb} + V_{td}V_{tb} = 0,$$

which can be represented as a triangle as shown in Fig. 65. The area of all the unitarity triangles is $\frac{1}{2}J$, where $J$ is the Jarlskog invariant, a convention-independent measure of CP-violation,

$$J = \text{Im}\{V_{ud}V_{us}V_{cb}V_{ub}^*\}.$$  

There are a large number of measurements which constrain the parameters in the unitarity triangle. They all measure different combinations of the parameters and over-constrain the location of the vertex of the unitarity triangle.

The magnitudes of the CKM elements control the lengths of the sides:

1. $|V_{ud}|$ is accurately measured in nuclear beta decay;
2. $|V_{cd}|$ can be measured using either semi-leptonic charm meson decays or using neutrino DIS cross sections;
3. $|V_{ub}|$ is measured using inclusive and exclusive semi-leptonic B meson decays to light mesons $B \rightarrow X_u\ell\bar{\nu}$ or $B \rightarrow \pi\ell\bar{\nu}$;
4. $|V_{cb}|$ is measured using inclusive and exclusive semi-leptonic B meson decays to charm mesons $B \rightarrow X_C\ell\bar{\nu}$ or $B \rightarrow D\ell\bar{\nu}$.

The CKM matrix elements which give the length of the remaining side can only be measured in loop-mediated processes. The most important of these, FCNCs, have already been discussed in the context of BSM physics in Section 9.1.4. These also gives rise to $B - \bar{B}$ mixing and oscillations, via the Feynman diagrams shown in Fig. 66.
The oscillation probability is

\[ P_{\text{oscillation}} = e^{-\Gamma t} \left[ \cosh \left( \frac{\Delta \Gamma t}{2} \right) + \cos \left( \Delta m t \right) \right], \tag{141} \]

where \( \Gamma \) is the average width of the mesons, \( \Delta \Gamma \) is the width difference between the mesons and \( \Delta m \) is the mass difference of the mesons. For both \( B_d \) and \( B_s \) mesons the \( \Delta m \) term dominates. From the box diagram

\[ \Delta m_q = -\frac{G_F^2 m_q^2 \eta_B m_{B_q} B_1 f_{B_q}^2}{6\pi^2} \frac{m_{W}^2}{m_{W}^2} \left( V_{tb}^* V_{td} \right)^2. \tag{142} \]

The decay constant \( f_{B_q} \) can be measured from leptonic decays \( B_q \to \ell^+ \nu_\ell \) but \( B_{B_q} \) comes from lattice QCD results. The QCD correction \( \eta_B \sim O(1) \).

The B-factories have studied \( B^0 - \bar{B}^0 \) mixing in great detail giving

\[ \Delta m_d = 0.507 \pm 0.005 \text{ps}^{-1}. \tag{143} \]

It is important to measure both \( B_d - \bar{B}_d \) and \( B_s - \bar{B}_s \) mixing as some hadronic uncertainties cancel in the ratio. The rate is \( \propto |V_{ts} V_{tb}^*|^2 \) due to the GIM mechanism. However, the high oscillation frequency makes \( B_s - \bar{B}_s \) mixing tricky to observe. The Tevatron observation relied on tagging the flavour of the B meson at production by observing an associated kaon from the fragmentation. The final result is

\[ \Delta m_s = 17.77 \pm 0.10 \text{(stat)} \pm 0.07 \text{(sys)}, \tag{144} \]

\[ |V_{td}|/|V_{ts}| = 0.2060 \pm 0.0007 \text{(exp)} \pm 0.008 \text{(theo)}. \]

The only source of CP-violation in the Standard Model is the complex phase in the CKM matrix. In order to see any effect we need at least two diagrams for the process with different CP-phases. There are three possibilities: CP-violation in the decay (direct); CP-violating in the mixing (indirect); CP-violation in the interference between decay and mixing. Example amplitudes are shown in Fig. 67.

The simplest type of CP-violation is direct CP-violation. This is the only possible type of CP-violation for charged mesons and is usually observed by measuring an asymmetry

\[ A_{f^\pm} \equiv \frac{\Gamma(M^- \to f^-) - \Gamma(M^+ \to f^+)}{\Gamma(M^- \to f^-) + \Gamma(M^+ \to f^+)} \text{ CP conserved} = 0. \tag{145} \]

If CP-symmetry holds, then \( |K_L| = \frac{1}{\sqrt{2}} (|K^0| + |\bar{K}^0|) \) would be a CP-eigenstate with \( |K_L| = |\bar{K}_L| \). If we take \( |M| = |K_L| \) and \( |f| = |\pi^- e^+ \nu_e| \) the corresponding CP-asymmetry is \( A_{\text{CP}} = (0.327 \pm 0.012)\% \), which means that \( K_L \) is not a CP-eigenstate.
and there is CP-violation. There are many possible modes which measure different combinations of the angles in the unitarity triangle. The observed flavour and CP-violation is consistent with the Standard Model, i.e. the description by the CKM matrix, see Fig. 68.

There is one final area of flavour physics which is important. The matter in the universe consists of particles and not antiparticles. There are three Sakharov conditions required for this to happen:

1. baryon number violation;
2. C-symmetry and CP-symmetry violation;
3. interactions out of thermal equilibrium.

There are non-perturbative effects in the SM which violate baryon number. However, the amount of CP-violation in the quark sector is not enough to give the observed matter-antimatter asymmetry, there might be more in the lepton sector, otherwise we need a new physics source of CP-violation.

C Color algebra

The color factors $C_F$ and $C_A$ correspond to the factors one gets for emitting a gluon off a quark or gluon line respectively.

\[ \begin{array}{c}
\begin{array}{c}
\vcenter{\hbox{\includegraphics[width=0.2\textwidth]{diagram1}}}
\end{array}
\end{array} = N_c C_F \]
\[ \begin{array}{c}
\begin{array}{c}
\vcenter{\hbox{\includegraphics[width=0.2\textwidth]{diagram2}}}
\end{array}
\end{array} = (N_c^2 - 1) C_A \]

The color factor for the splitting of a gluon into a quark-antiquark pair is given by $T_R$. 

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Figure 68: Experimental measurement of the unitarity triangle taken from Ref. [8].

\[ \begin{array}{c}
\text{Figure 68: Experimental measurement of the unitarity triangle taken from Ref. [8].}
\end{array} \]

One can compute color factors using a set of pictorial rules (see [36] for more details.)
All these rules follow from the properties of the SU(3) color group.

\[ T_{ij}^a T_{jk}^a = \frac{1}{2} \left( \delta_{il} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right) \]

\[ \text{The three-gluon vertex can be rewritten as:} \]

\[ i f^{abc} = 2 \left( \text{Tr} \left[ T^a T^b T^c \right] - \text{Tr} \left[ T^a T^c T^b \right] \right) \]
Here is an example of a calculation of a color factor with the pictorial method.

\[
\left( \begin{array}{c}
\text{fermion loop} \\
\end{array} \right) = \left( \begin{array}{c}
\text{gluon loop}
\end{array} \right) = \frac{1}{2} \left[ \frac{N_c^2}{N_c} - \frac{1}{N_c} \right] = N_c \left( N_c^2 - 1 \right)
\]

We have used the fact that a closed fermion loop with no gluon attachments amounts to a factor of \(N_c\), while a closed gluon loop would give a factor of \(N_c^2 - 1\).

A gluon loop on a gluon line can be written as the same line without the loop but with a factor of \(N_c\).

References


DARK MATTER

Dr David G. Cerdeño (IPPP, University of Durham)